Reviewing classical interpretations of Cantor's, Gödel's, Tarski's, and Turing's reasoning

(*and addressing some grey areas in the foundations of mathematics, logic and computability*)

Blog page: *Foundations of Mathematics, Logic & Computability*

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"*We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on. So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work.*"

. . . Richard P. Feynman, in his Nobel Lecture, 1966

"*If you ask a philosopher what the main problems are in the philosophy of mathematics, then the following two are likely to come up: what is the status of mathematical truth, and what is the nature of mathematical objects? That is, what gives mathematical statements their aura of infallibility, and what on earth are these statements about?*"

. . . W. T. Gowers in his talk, "*Does mathematics need a philosophy?*", presented before the Cambridge University Society for the Philosophy of Mathematics and Mathematical Sciences, 2002.

"*Let not posterity judge us as having spent our lives polishing the pebbles and tarnishing the diamonds.*"

...Unknown (In memory of my friend and mentor: Ashok Chadha).

Blog page: *Let not posterity judge us as having spent our lives polishing the pebbles and tarnishing the diamonds*

1 Preamble

(July 2005): Standard interpretations of the formal reasoning, and conclusions, of classical, first order, theory—based primarily on the work of Cantor, Gödel, Tarski, and Turing—argue that the truth (satisfiability) of the propositions of a formal mathematical language, under an interpretation, is, both, non-algorithmic and essentially unverifiable constructively.

In these investigations, we argue (April 2002 onwards) that—if mathematics is to serve as a universal set of languages of, both, precise expression and unambiguous communication—such interpretations may need to be balanced by an alternative, constructive and intuitionistically unobjectionable, interpretation—of classical foundational concepts—in which truth (satisfiability) is defined effectively.

2 The philosophical and mathematical significance of Aristotle's particularisation in the foundations of mathematics, logic and computability

2.1 The Factorisation of all $m \leq n$ needs $O(1)\sum_{i=1}^{n}(\sqrt{i}/log_e i)$ arithmetical operations

. . . ***(August 2012)***

We define a modular Eratosthenes matrix that computes all the divisors of n in $O(1)\sum_{i=1}^{n} i$ bounded arithmetical operations. We then show that $O(1)\sum_{i=1}^{n}\pi(\sqrt{i})$ bounded arithmetical operations are necessary and sufficient to compute all the prime divisors of all $m \leq n$.

2.2 The computational complexity classes *Algorithmically Verifiable* and *Algorithmically Computable*

. . . ***(April 2014)***

We define the computational complexity classes AV (algorithmically verifiable) and AC (algorithmically computable) from a *finitary*, evidence-based, number-theoretic perspective which is *strictly* grounded within the classical first-order logic FOL. The finitary perspective is rooted in the recognition, first, that the classical Tarskian assignment of satisfaction and truth values to the formulas of a first-order Peano Arithmetic such as PA—under which the PA Axioms interpret as true and the PA Rules of Inference preserve such truth—can be defined in two distinctly different ways: (a) *non-finitarily* in terms of algorithmic verifiability, and (b) *finitarily* in terms of algorithmic computability; and, second, that not every algorithmically true arithmetical relation is algorithmically computable. From this perspective it appears that standard definitions of the $PvNP$ problem may be implicitly assuming set-theoretically (*non-finitarily*)—without a corresponding number-theoretic (*finitary*) foundation—that every propositional formula which is algorithmically verifiable *in polynomial time* is necessarily algorithmically computable. However, we argue that such an assumption is logically fragile, since we can define arithmetic formulas which are algorithmically verifiable—hence in AV—but not algorithmically computable—whence not in AC.

Blog page: *The $PvNP$ Separation Problem*

2.3 An arithmetical perspective on Cantor's Continuum Hypothesis

. . . ***(September 2013)***

We distinguish between algorithmically verifiable functions and algorithmically computable functions. We then show that Gödel's β-function uniquely corresponds each real number to an algorithmically verifiable arithmetical function. We conclude that there is no arithmetically definable set whose cardinality is strictly between the cardinality $\aleph_0$ of the integers and the cardinality $2^{\aleph_0}$ of the real numbers. We further conclude that although the set-theoretical interpretation of the Continuum Hypothesis is not provable from the axioms of ZF, the arithmetical interpretation of the Continuum Hypothesis is provable from the axioms of PA.

Blog page: The answer that Hilbert probably would not have foreseen for the first of his twenty three problems!

2.4 Why some standard arguments for the existence of non-standard models of the first-order Peano Arithmetic PA that contain elements other than the natural numbers are logically fragile . . . *(May 2013)*

We show why two standard arguments for the existence of non-standard models of the first-order Peano Arithmetic PA are logically fragile. We then show that Gödel's argument for the existence of a non-standard model of PA does yield a model of PA other than the standard model, but we cannot conclude that the domain is other than the domain $\mathbb{N}$ of the natural numbers unless we make the non-constructive—and logically fragile—assumption that PA is ω-consistent.

Blog page: The case against non-standard models of PA

2.5 Three perspectives of FOL: Classical, Intuitionist and Universal . . . *(March 2013)*

In this investigation we highlight two Meta-theses that essentially determine the differences between Classical applications of first order logic and Intuitionist applications of first order logic. We indicate a Universal perspective that discards the extreme postulations which prevent the universal applicability of both these Meta-theses. We then consider some consequences of the application of first order logic under a Universal Meta-thesis to the first order Peano Arithmetic PA under which: (a) PA is consistent but not ω-consistent; (b) The provable formulas of PA are precisely those that are algorithmically computable as always true under a sound interpretation of PA; (c) PA is categorical; and (d) The Standard interpretation of PA is not sound.

(***Presentation***)

2.6 Aristotle's particularisation: The Achilles' heel of Hilbertian and Brouwerian perspectives of classical logic . . . *(November 2012)*

We take the view that the defining beliefs of both the Hilbertian and the Brouwerian perspectives of classical logic have been unreasonably influenced by their uncritical acceptance in the first case, and their uncritical denial in the second, of Aristotle's particularisation. This is the postulation that an existentially quantified formula—such as '$[(\exists x)P(x)]$'—of a first order language S can be assumed to always interpret as the proposition, 'There exists some s in the domain D of the interpretation such that $P^*(s)$ holds in D', without inviting inconsistency. We show that if the first order Peano Arithmetic PA is consistent, then the postulation is false. However, we cannot conclude from this that the Law of the Excluded Middle, too, is false.

(Presented on 6^{th} April at the session on the '*Scope of Logic Theorems*' at UNILOG'2013, 4^{th} World Congress and School on Universal Logic, 29^{th} March 2013 - 7^{th} April 2013, Rio de Janeiro, Brazil.)

2.7 A suggested mathematical perspective for the EPR argument . . . *(October 2012)*

We suggest that the paradoxical element in the EPR argument may reflect the absence of appropriately defined mathematical functions that can represent the deterministic yet unpredictable characteristics of quantum behaviour. The anomaly may disappear if a physicist could cogently argue that: (i) All properties of physical reality can be deterministic, in the sense that any physical property can have one, and only one, value at any time $t(n)$, where the value is completely determined by some natural law which need not, however, be algorithmic. (ii) There are elements of such a physical reality whose properties at any time $t(n)$ are determined completely in terms of their putative properties at some earlier time $t(0)$. Such properties are representable mathematically by algorithmically computable functions. The Laws of Classical Mechanics describe the nature and behaviour of such physical reality only. (iii) There can be elements of such a physical reality whose properties at any time $t(n)$ cannot be theoretically determined completely from their putative properties at some earlier time $t(0)$. Such properties are only representable mathematically by algorithmically verifiable, but not algorithmically computable, functions. The Laws of Quantum Mechanics describe the nature and behaviour of such physical reality.

(Presented on 7^{th} April at the workshop on '*Logical Quantum Structures*' at UNILOG'2013, 4^{th} World Congress and School on Universal Logic, 29^{th} March 2013 - 7^{th} April 2013, Rio de Janeiro, Brazil.)

Blog page: *A suggested mathematical perspective for the EPR argument*

2.8 Paper: Evidence-Based Interpretations of PA . . . *(March 2012)*

Presentation: Evidence-Based Interpretations of PA . . . *(July 2012)*

We show that Tarski's inductive definitions admit evidence-based interpretations of the first-order Peano Arithmetic PA that allow us to define the satisfaction and truth of the quantified formulas of PA *constructively* over the domain $\mathbb{N}$ of the natural numbers in *two* essentially different ways: (a) in terms of algorithmic verifiabilty; and (b) in terms of algorithmic computability. We argue that the algorithmically computable PA-formulas *can* provide a finitary interpretation of PA from which we may conclude that PA is consistent in an intuitionistically unobjectionable manner.

(Proceedings of the Computational Philosophy Symposium at the AISB/IACAP World Congress 2012 - Alan Turing Centenary 2012, University of Birmingham, Birmingham UK, July 2^{nd} to 6^{th}, 2012. Papers #2.5 and #2.12 extend the arguments of this paper.)

2.9 Meeting Wittgenstein's requirement of 'truth' in Gödel's formal reasoning . . . *(January 2012)*

In this short note we argue that Wittgenstein's objection to Gödel's interpretation of his (Gödel's) incompleteness theorems was justified since all mathematically true (even if formally unprovable) assertions necessarily follow *formally* from the axioms and rules of inference of the corresponding formal mathematical language.

(Draft)

2.10 A foundational perspective on the semantic and logical Paradoxes . . . *(January 2012)*

We argue that, from a foundational perspective, the paradoxical element in the familiar semantic and logical paradoxes is simply a reflection of the attempt to ask of a language more than it is designed to deliver.

Blog page: *A foundational perspective on the semantic and logical paradoxes*

2.11 Is Gödel's undecidable proposition an 'ad hoc' anomaly? . . . *(July 2011)*

We show that if the standard interpretation of PA is sound, then Gödel's arithmetical representation of any recursive relation yields an undecidable PA formula. This Gödelian characteristic is merely a reflection of the fact that, by the instantiational nature of their constructive definition in terms of Gödel's β-function, such formulas are designed to be algorithmically verifiable, but not algorithmically computable, under the standard interpretation of PA.

Blog page: *Is Gödel's undecidable proposition an 'ad hoc' anomaly?*

2.12 Some consequences of interpreting the associated logic of a first-order Peano Arithmetic PA finitarily . . . *(July 2011)*

We show that Tarski's inductive definitions allow us to define the satisfaction and truth of the formulas of the first-order Peano Arithmetic PA *constructively* over the domain $\mathbb{N}$ of the natural numbers in *two* ways: (a) in terms of algorithmic verifiabilty; and (b) in terms of algorithmic computability. We show that the standard interpretation of PA defines the satisfaction and truth of the formulas of the first-order Peano Arithmetic PA constructively in terms of algorithmic verifiability. It is accepted that this interpretation cannot lay claim to be finitary, since it does not lead to a finitary justification of the finite Axiom Schema of Induction of PA from which we may conclude—in an intuitionistically unobjectionable manner—that PA is consistent. However, we now show that the PA-axioms—including the Axiom Schema of (finite) Induction—are algorithmically computable as satisfied / true under the standard interpretation of PA, and that the PA rules of inference preserve algorithmically computable satisfiability / truth under the standard interpretation. We conclude that the algorithmically computable PA-formulas can provide a finitary interpretation of PA from which we may conclude that PA is consistent in an intuitionistically unobjectionable manner. We define such an interpretation and show that, if the associated logic is interpreted finitarily then (i) PA is categorical and (ii) Gödel's Theorem VI holds vacuously in PA since PA is consistent

but not ω-consistent. This reflects the fact that PA is ω-consistent if, and only if, Aristotle's particularisation is presumed to always hold under any interpretation of the associated logic; and that the standard interpretation of PA is a model of PA if, and only if, PA is ω-consistent.
(Draft)

2.13 A foundational argument for weakening the Church-Turing Thesis . . . *(February 2011)*

We conclude from Gödel's Theorem VII of his seminal 1931 paper that every recursive function $f(x_1, x_2)$ is representable in the first-order Peano Arithmetic PA by a formula $[F(x_1, x_2, x_3)]$ which is algorithmically verifiable, but not algorithmically computable, *if* we assume that the negation of a universally quantified formula of the first-order predicate calculus is always indicative of the existence of a counter-example under the standard interpretation of PA. We conclude that the standard postulation of the Church-Turing Thesis does not hold if we define a number-theoretic formula as effectively computable if, and only if, it is algorithmically verifiable; and needs to be replaced by a weaker postulation of the Thesis as an equivalence.
(Draft)

2.14 The case against the ordinal-based proof of Goodstein's Theorem . . . *(February 2011)*

The ordinal-based argument for Goodstein's Theorem meets William Gasarch's criteria of an argument that prima facie defies belief. We show that the disbelief is justified since Goodstein's argument can be carried out completely over the structure of the natural numbers without appealing to any properties of transfinite ordinal sequences. However the arithmetical argument does not support the conclusion that every Goodstein sequence over the natural numbers must terminate finitely. We show that the ordinal-based argument for Goodstein's Theorem is a striking case of proving a Theorem in ZF and *implicitly* postulating that it must be provable in PA.

Blog page: *The case against Goodstein's Theorem*

2.15 Gödel's Theorem VII and the PvNP problem . . . *(February 2011)*

We conclude from Gödel's Theorem VII of his seminal 1931 paper that every recursive function $f(x_1, x_2)$ is representable in the first-order Peano Arithmetic PA by a formula $[F(x_1, x_2, x_3)]$ which is algorithmically verifiable, but not algorithmically computable, *if* we assume that the negation of a universally quantified formula of the first-order predicate calculus is always indicative of the existence of a counter-example under the standard interpretation of PA. We consider the possible significance of this for expressing and conditionally resolving the PvNP problem.
(Draft)

2.16 Does resolving PvNP require a paradigm shift? . . . *(November 2009)*

We shall argue that a resolution of the PvNP problem requires building an iff bridge between the domain of provability and that of computability. The former concerns how a human intelligence decides the truth of number-theoretic relations, and is formalised by the first-order Peano Arithmetic PA following Dededekind's axiomatisation of Peano's Postulates. The latter concerns how a human intelligence computes the values of number-theoretic functions, and is formalised by the operations of a Turing Machine following Turing's analysis of computable functions. We shall show that such a bridge requires objective definitions of both an 'algorithmic' interpretation of PA, and an 'instantiational' interpretation of PA. We shall show that both interpretations are implicit in the definition of the—hitherto subjectively defined—'standard' interpretation of PA. However the existence of, and distinction between, the two constructively definable interpretations—and the fact that the former is sound whilst the latter is not—is obscured by the extraneous presumption under the 'standard' interpretation of PA that Aristotle's particularisation must hold over the structure $\mathbb{N}$ of the natural numbers. We shall argue that recognising the falseness of this belief awaits a paradigm shift in our perception of the 'standard' interpretation of classical first-order logic, and its applicability—under Tarski's analysis of the concept of truth in the languages of the deductive sciences—to the 'standard' interpretation of PA. We shall then show that an arithmetical formula $[F]$ is PA-provable if, and only if, $[F]$ interprets as true under an algorithmic interpretation of PA. We shall finally show how it then follows from Gödel's construction of a formally 'undecidable' arithmetical proposition that there is a Halting-type PA formula which—by Tarski's definitions—is algorithmically verifiable as true, but not algorithmically computable as true, under a sound interpretation of PA.

(Draft)

2.17 Presentation: Is there a danger to humankind in actively seeking out an extra-terrestrial intelligence? . . . *(July 2009)*

We address the question raised by Carl Sagan in his 1985 novel 'Contact': Is there a danger to humankind in actively seeking out an extra-terrestrial intelligence? We conclude that any ETI which is capable of a sound interpretation of transmissions expressed in the language of the Peano Arithmetic, PA, will *not* 'see' us as an essentially different—hence possibly threatening—'intelligence'.

(Presented on 14th July at the 2009 International Conference on Theoretical and Mathematical Foundations of Computer Science, Orlando, Florida, USA; and on 16th July 2009 at the 2009 International Conference on Foundations of Computer Science, Las Vegas, Nevada, USA)

2.18 Turing and a sound, finitary, interpretation of PA . . . *(March 2009)*

We define a sound, finitary, interpretation of PA, and show that PA is categorical.

(Proceedings of the 2009 International Conference on Theoretical and Mathematical Foundations of Computer Science, 13-16 July 2009, Orlando, Florida, USA)

2.19 Rosser and formally undecidable arithmetical propositions . . . *(March 2009)*

We review Rosser's claim that Gödel's reasoning can be recast to arrive at his intended result without the assumption of ω-consistency. We show that Rosser's argument appeals to a fundamental tenet of this logic—namely Aristotlean particularisation—which implies ω-consistency.

(Proceedings of the 2009 International Conference on Foundations of Computer Science, 13-16 July, Las Vegas, Nevada, USA)

Blog page: *Can someone tell me what is so special about Rossers proof of formally undecidable arithmetical propositions?*

2.20 Gödel and formally undecidable arithmetical propositions . . . *(March 2009)*

We show that any model (such as the 'standard' model) of PA which appeals to Aristotlean particularisation—under Tarski's definitions of what constitute's an interpretation of the language—is not sound.

(Proceedings of the 2009 International Conference on Theoretical and Mathematical Foundations of Computer Science, 13-16 July, Orlando, Florida, USA)

2.21 Cohen and the Axiom of Choice . . . *(March 2009)*

The logic underlying our current interpretations of all first-order formal languages—which provide the formal foundations for all computing languages—is Aristotle's logic of predicates. We show, first, that a fundamental tenet of this logic, namely Aristotlean particularisation, is a subjective, and objectively unverifiable, postulation that is 'stronger' than the Axiom of Choice; and that, second, any putative model of ZF which appeals to Aristotlean particularisation—under Tarski's definitions of what constitute's an interpretation of the language—is not sound.

(Proceedings of the 2009 International Conference on Foundations of Computer Science, 13-16 July, Las Vegas, Nevada, USA)

Blog page: *Placing Cohens proof of the Independence of the Axiom of Choice in perspective*

2.22 A case for reviewing current interpretations of Cantor's, Gödel's, Turing's and Tarski's formal reasoning . . . *(August 2008)*

We argue that interpretations of Cantor's, Gödel's, Turing's and Tarski's formal results—which are viewed as building upon each other chronologically in current expositions of classical theory—need to be viewed afresh from a perspective of logical entailment, rather than from the perspective of historical happenstance that they presently reflect.

(Draft)

3 Research Directions

(Except where indicated otherwise, the following investigations have not been peer-reviewed. The author welcomes any comments and feedback - whether on the contents, presentation, or reading and printing problems. E-mail: anandbATvsnl.com)

3.1 Why we need not accept limits on provability and effective computation imposed by current interpretations of Gödel's reasoning and the Church-Turing Thesis . . . *(August 2008)*

We show that first-order Peano Arithmetic is algorithmically complete; that the Church and Turing Theses are refutable; and that, if ZF is consistent, then the Halting problem is effectively solvable by a Turing-machine.

(Draft)

3.2 The significance of Feynman's cover-up factor . . . *(July 2008)*

We argue that Gödel's interpretation, and assessment, of his own formal reasoning in his seminal 1931 paper on formally undecidable arithmetical propositions is, essentially, a post-facto imposition that continues to echo in, and influence, standard expositions of Gödel's reasoning misleadingly.

(Draft)

3.3 Why we cannot apply set-theoretical arguments to number theory unrestrictedly . . . *(July 2008)*

We critically review the application—to the first-order number-theory PA—of set-theoretical arguments which admit the conclusions, first, that PA has non-standard models and, second, that every Goodstein sequence defined over the natural numbers must terminate finitely.

(Draft)

3.4 A trivial solution to the PvNP problem . . . *(May 2008)*

We show that Gödel has defined an arithmetical relation $R(n)$ which—when treated as a Boolean function—is constructively computable as true for any given natural number n, but which is not Turing-computable as true for any given natural number n. This implies that the current formulation of the PvNP problem admits a trivial logical solution that is not significant computationally.

(Proceedings of the 2008 International Conference on Foundations of Computer Science, July 14-17, 2008, Las Vegas, USA. This presentation may differ in formatting and other inessential details from the formal paper.)

3.5 A finitary model of Peano Arithmetic . . . *(May 2008)*

We define a finitary model of first-order Peano Arithmetic in which satisfaction and quantification are interpreted constructively in terms of Turing-computability.

(Proceedings of the 2008 International Conference on Foundations of Computer Science, July 14-17, 2008, Las Vegas, USA. This presentation may differ in formatting and other inessential details from the formal paper)

3.6 Why Brouwer was justified in his objection to Hilbert's unqualified interpretation of quantification . . . *(April 2008)*

We define a finitary model of first-order Peano Arithmetic in which quantification is interpreted constructively in terms of Turing-computability, and show that it is inconsistent with the standard interpretation of PA.

(Proceedings of the 2008 International Conference on Foundations of Computer Science, July 14-17, 2008, Las Vegas, USA. This presentation may differ in formatting and other inessential details from the formal paper)

Blog page: *Why Brouwer was justified in his objection to Hilberts unqualified interpretation of quantification*

4 Research outlines

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4.1 The case against Goodstein's Theorem . . . *(February 2008)*

4.2 Can we really falsify truth by dictat? . . . *(January 2008)*

(This presentation may differ in minor typographical and other details from that published as an on-going research outline in the January 2008 issue of The Reasoner, Vol(2)1 p7-8)

4.3 A constructive definition of the intuitive truth of the Axioms and Rules of Inference of Peano Arithmetic . . . *(December 2007)*

(This presentation may differ in minor typographical and other details from that published as an on-going research outline in the December 2007 issue of The Reasoner, Vol(1)8 p6-7)

4.4 Why we shouldn't fault Lucas and Penrose for continuing to believe in the Gödelian argument against computationalism - II . . . . . . *(November 2007)*

(This presentation may differ in minor typographical and other details from that published as an on-going research outline in the November 2007 issue of The Reasoner, Vol(1)7 p2-3)

4.5 Why we shouldn't fault Lucas and Penrose for continuing to believe in the Gödelian argument against computationalism - I . . . *(October 2007)*

(This presentation may differ in minor typographical and other details from that published as an on-going research outline in the October 2007 issue of The Reasoner, Vol(1)6, p3-4)

4.6 The Mechanist's Challenge - Did we really hope to get away with The Gödelian argument? . . . *(September 2007)*

(This presentation may differ in minor typographical and other details from that published as an on-going research outline in the September 2007 issue of The Reasoner, Vol(1)5 p5-6)

5 Preliminary Investigations

(Except where indicated otherwise, the following investigations have not been peer-reviewed. The author welcomes any comments and feedback - whether on the contents, presentation, or reading and printing problems. E-mail: anandbATvsnl.com)

5.1 Two presumptions in Gödel's interpretation of his own, formal, reasoning that are classically objectionable . . . *(March 2007)*

Standard expositions of Gödel's 1931 paper on undecidable arithmetical propositions are based on two presumptions in Gödel's 1931 interpretation of his own, formal, reasoning - one each in Theorem VI and in Theorem XI - which do not meet Gödel's, explicitly stated, requirement of classically constructive, and intuitionistically unobjectionable, reasoning. We see how these objections can be addressed, and note some consequences.

5.2 An elementary proof that P $\neq$ NP . . . *(February 2007)*

We show that, if PA has no non-standard models, then P $\neq$ NP. We then give an elementary proof that PA has no non-standard models.

5.3 Why we shouldn't fault Lucas and Penrose for continuing to believe in the Gödelian argument against computationalism . . . *(July 2006)*

The only fault we can fairly lay at Lucas' and Penrose's doors, for continuing to believe in the essential soundness of the Gödelian argument, is their naive faith in, first, non-verifiable assertions in standard expositions of classical theory, and, second, in Gödel's unvalidated interpretation of his own formal reasoning. We show why their faith is misplaced in both instances.

5.4 P $\neq$ NP under a constructive interpretation of Peano Arithmetic . . . *(March 2006)*

We show that all provable arithmetical formulas are Turing-decidable under the standard interpretation of a standard, first-order, Peano Arithmetic, PA. An immediate consequence is that the set of Gödel-formulas of PA is empty, and that PA has no non-standard models. We show that this implies, further, that P $\neq$ NP.

5.5 Why Brouwer was right in suggesting that Hilbert's Law of The Excluded Middle needed qualification . . . *(March 2006)*

We reproduce Hilbert's axiomatic formalisation of Number Theory, and argue that his enunciation of the Law of the Excluded Middle is inconsistent with a Turing-verifiable model of the axioms under the standard interpretation.

5.6 Naïve Philosophical Foundations . . . *(December 2005)*

This 1967 soliloquy outlines some naïve philosophical arguments underlying the thesis that mathematics ought to be viewed simply as a universal set of languages, some of precise expression, an

5.7 From *perfect* numbers to a generating function for the unrestricted factorisations of an integer . . . *(August 2011. Transcript of a 1964 note updated on 11/03/1993.)*

Unusual connections between additive and multiplicative number theory, with roots in a trivial multiplicative analogue for the Grecian *perfect* numbers.

5.8 A Minimal Prime Generating Theorem that suggests the Prime Difference is $O(\pi(p(n)^{1/2}))$ . . . *(December 2005)*

Although this 1964 paper is not, strictly, a foundational paper, it developed as an off-shoot to the foundational query: Do we discover the natural numbers (Platonically), or do we construct them linguistically? The paper also assumes computational significance in the light of Agrawal, Kayal and Saxena's August 2002 paper, "PRIMES is in P", since both the Trim and Compact Number Generating algorithms - each of which generates all the primes - are deterministic algorithms that suggest the Prime Difference, $d_p(n)$, is $O(\pi(p(n)^{1/2}))$.

5.9 PA is instantiationally and arithmetically complete, but algorithmically incomplete: An alternative interpretation of Gödelian incompleteness under Church's Thesis that links formal logic and computability . . . *(July 2005)*

We define instantiational and algorithmic completeness for a formal language, and arithmetical completeness for Peano Arithmetic. We show that, in the presence of Church's Thesis, an alternative interpretation of Gödelian incompleteness is that Peano Arithmetic is instantiationally complete, but algorithmically incomplete. We then postulate a Provability Thesis that links Peano Arithmetic and effective algorithmic computability - just as Church's Thesis links Recursive Arithmetic and effective instantiational computability - under which PA is arithmetically complete.

5.10 The Göbbellian Syndrome . . . *(June 2005)*

Can we really falsify truth by dictat? A critical note on J. R Lucas' 1996 remarks concerning non-standard models of first order Peano Arithmetic.

5.11 The Mechanist's Challenge . . . *(June 2005)*

Did we really hope to get away with The Gödelian Argument? A critical response to J. R Lucas' 1996 articulation of his 1961 argument.

5.12 An arguable inconsistency in ZF . . . *(February 2005)*

Classical theory proves that every primitive recursive function is strongly representable in PA; that formal Peano Arithmetic, PA, and formal primitive recursive arithmetic, PRA, can both be interpreted in Zermelo-Fraenkel Set Theory, ZF; and that if ZF is consistent, then PA+PRA is consistent. However it is silent on the consistency of the latter. We now show that PA+PRA is, in fact, inconsistent; hence ZF, too, is inconsistent.

5.13 An arguable addition to the standard Deduction Theorems of first-order theories . . . *(September 2004)*

We consider the immediate consequence of an arguable addition to the standard Deduction Theorems of first order theories.

(This is a formal presentation of the arguments of an earlier essay)

5.14 Is the Halting problem effectively solvable non-algorithmically? . . . *(September 2004)*

We consider a perspective for the belief that there are classically two equivalent ways to look at the mathematical notion of proof: logical, as a finite sequence of sentences strictly obeying some axioms and inference rules; and computational, as a specific type of computation. We then consider a weaker Arithmetical Provability thesis which implies that the Church and Turing Theses are false when expressed as identities, and that the Halting Problem is effectively solvable, albeit non-algorithmically.

. . . *(April 2012)*

We consider an intuitively plausible Arithmetical Provability thesis which implies that the Halting Problem is effectively solvable, albeit non-algorithmically.

(This presents the main argument only.)

5.15 Do Gödel's incompleteness theorems set absolute limits on the ability of the brain to express and communicate mental concepts verifiably? . . . *(June 2004)*

Classical interpretations of Gödel's formal reasoning, and of his conclusions, implicitly imply that mathematical languages are essentially incomplete, in the sense that the truth of some arithmetical propositions of any formal mathematical language, under any interpretation, is both non-algorithmic and essentially unverifiable. However, a language of general scientific discourse, which intends to mathematically express and unambiguously communicate intuitive concepts that correspond to scientific investigations, cannot allow its mathematical propositions to be interpreted ambiguously. Such a language must, therefore, define mathematical truth verifiably. We consider a constructive interpretation of classical Tarskian truth, and of Gödel's reasoning, under which any formal system of Peano Arithmetic - classically accepted as the foundation of all our mathematical languages - is verifiably complete in the above sense. We show how some paradoxical concepts of Quantum mechanics can, then, be expressed, and interpreted, naturally under a constructive definition of mathematical truth.

(This is an update of the essay published in the invited article section of the OA Web Journal NeuroQuantology 2004; 2: 60-100)

5.16 How definitive is the standard interpretation of Goodstein's argument? . . . *(Edited version with improved notation: April 2011)*

Goodstein's argument is, essentially, that the hereditary representation, $m_{[b]}$, of any given natural number m in the natural number base b, can be mirrored in Cantor Arithmetic, and used to well-define a finite, decreasing, sequence of transfinite ordinals, each of which is not smaller than the ordinal corresponding to the corresponding member of Goodstein's sequence of natural numbers $G(m)$. The standard interpretation of this argument is, first, that $G(m)$ must, therefore, converge; and, second, that this conclusion - Goodstein's Theorem - is unprovable in Peano Arithmetic, but true under its standard interpretation. We argue however that even assuming Goodstein's Theorem is indeed unprovable in PA, its truth must, nevertheless, be an intuitionistically unobjectionable consequence of some constructive interpretation of Goodstein's reasoning. We consider such an interpretation, and highlight why the standard interpretation of Goodstein's argument ought not to be accepted as definitive.

. . . *(November 2003)*

5.17 The Einstein-Bohr debate: Can Laplace's formula model a deterministic universe that is irreducibly probabilistic? . . . *(June 2003)*

If we assume the Thesis that any classical Turing machine T which halts on every n-ary sequence of natural numbers as input, in a determinate time $t(n)$, determines a PA-provable formula whose standard interpretation is an n-ary arithmetical relation $f(x_1, \ldots, x_n)$ that holds if and only if

T halts, then we can define Laplace's formula recursively such that it can express the state of a deterministic quantum universe which is irreducibly probabilistic.

5.18 How definitive is the standard interpretation of Gödel's Incompleteness Theorem? . . . *(June 2003)*

Standard interpretations of Gödel's "undecidable" proposition $[(Ax)R(x)]$ argue that, although $[\neg(Ax)R(x)]$ is PA-provable if $[(Ax)R(x)]$ is PA-provable, we may not conclude from this that $[\neg(Ax)R(x)]$ is PA-provable. We show that such interpretations are inconsistent with a standard Deduction Theorem of first order theories.

(We give a more formal presentation of this argument in a later essay)

5.19 Why we must heed Wittgenstein's "notorious paragraph" . . . *(May 2003)*

In this essay we argue that although Wittgenstein's reservations on Gödel's interpretation of his own formal reasoning are, indeed, of historical importance, the uneasiness that academicians and philosophers continue to sense, and express, over standard interpretations of Gödel's formal reasoning - even seventy years after the publication of his seminal 1931 paper - is of much greater significance and relevance to us today: such uneasiness should be seen as indicating specific points of possible ambiguity in foundational mathematical concepts that need to be addressed on philosophical grounds rather than dismissed on technicalities.

5.20 Is the Halting probability a Dedekind real number? . . . *(May 2003)*

In a recent historical overview, Cristian S. Calude, Elena Calude, and Solomon Marcus identify eight stages in the development of the concept of a mathematical proof in support of an ambitious conjecture: we can express classical mathematical concepts adequately only in a mathematical language in which both truth and provability are essentially unverifiable. In this essay we show first that the concepts underlying their thesis *can*, however, be interpreted constructively; and second that an implicit thesis in the authors' arguments implies that the probability of a given Turing machine halting on a given input cannot be expressed as a Dedekind real number.

5.21 Can we express every transfinite concept constructively? . . . *(May 2003)*

In a forthcoming book, professional computer scientist and physicist Paul Budnik presents an exposition of classical mathematical theory as the backdrop to an elegant thesis: we can interpret any model of a formal system of Peano Arithmetic in an appropriate digital computational language. In this essay we attempt - without addressing the question of whether or not Budnik succeeds in establishing his thesis convincingly - to identify dogmas of standard interpretations of classical mathematical theory that appear to be implicit in Budnik's exposition, and to correspond to them dogmas of a constructive interpretation of classical theory.

5.22 The formal roots of Platonism . . . *(April 2003)*

We present some arguments for the thesis that a set-theoretic inspired faith in the ability of intuitive truth to faithfully reflect relationships between elements of a Platonic universe may be as misplaced as an assumption that such truth cannot be expressed in a constructive, and effectively verifiable, manner.

5.23 Can Turing machines capture everything we can compute? . . . *(April 2003)*

If we define classical foundational concepts constructively and introduce non-algorithmic effective methods into classical mathematics, then we can bridge the chasm between truth and provability, and define computational methods that are not Turing computable.

5.24 Three beliefs that lend illusory legitimacy to Cantor's diagonal argument . . . *(April 2003)*

Whatever other beliefs there may remain for considering Cantor's diagonal argument as mathematically legitimate, there are three that, prima facie, lend it an illusory legitimacy; they need to be explicitly discounted appropriately. The first - Cantor's diagonal argument defines a non-countable Dedekind real number; the second - Gödel uses the argument to define a formally undecidable, but interpretively true, proposition; and the third - Turing uses the argument to define an uncomputable Dedekind real number.

5.25 Is there a "loophole" in Gödel's interpretation of his formal reasoning and its consequences? . . . *(April 2003)*

We formally define a "mathematical object" and "set". We then argue that expressions such as "$(\forall x)F(x)$", and "$(\exists x)F(x)$", in an interpretation M of a formal theory P, may be taken to mean "$F(x)$ is true for all x in M", and "$F(x)$ is true for some x in M", respectively, if, and only if, the predicate letter "F" is a mathematical object in P. In the absence of such a meta-proof, the expressions "$(\forall x)F(x)$", and "$(\exists x)F(x)$", can only be taken to mean that "$F(x)$ is true for any given x in M", and "It is not true that $F(x)$ is false for any given x in M", respectively, indicating that the predicate "$F(x)$" is well-defined, and effectively decidable individually, for any given value of x, but that there may be no uniform effective method (algorithm) for such decidability. We show how some paradoxical concepts of Quantum mechanics can then be expressed in a constructive interpretation of standard Peano's Arithmetic.

5.26 Is there a duality in the classical acceptance of non-constructive, foundational, concepts as axiomatic? . . . *(March 2003)*

We consider a philosophical question that is implicit in Bringsjord's paper "The Narrational Case Against Church's Thesis": Is there a duality in the classical acceptance of non-constructive foundational concepts as axiomatic?

5.27 Is a deterministic universe logically consistent with a probabilistic Quantum Theory? . . . *(December 2002)*

If we assume the Thesis that any classical Turing machine T which halts on every n-ary sequence of natural numbers as input determines a PA-provable formula, whose standard interpretation is an n-ary arithmetical relation $f(x_1, \ldots, x_n)$ that holds if and only if T halts, then standard PA can model the state of a deterministic universe that is consistent with a probabilistic Quantum Theory. Another significant consequence of this Thesis is that every partial recursive function can be effectively defined as total.

(This essay has been superceded by a later essay)

5.28 Are there parts of our arithmetical competence that no sound formal system can duplicate? . . . *(October 2002)*

In a review of Roger Penrose's "Shadows of the Mind", David Chalmers opined as implausible that there could be parts of our arithmetical competence that no sound formal system could ever duplicate. We prove, however, that the recursive number-theoretic relation $x = Sb(y\ 19|Z(y))$ - which is accepted as effectively verifiable since the recursive function $Sb(y\ 19|Z(y))$ is Turing-computable - cannot be introduced through definition in any consistent formal system of Arithmetic.

(This essay reproduces Meta-thesis 1 and other Meta-lemmas from an earlier essay)

5.29 Some consequences of defining mathematical objects constructively and mathematical truth effectively . . . *(October 2002)*

(*Previous title: Some consequences of a recursive number-theoretic relation that is not the standard interpretation of any of its formal expressions*)

Standard interpretations of classical first order theory - rooted primarily in the works of Cantor, Gödel, Tarski, and Turing - argue that the truth of the propositions of a formal mathematical language, under an interpretation, is non-algorithmic and essentially unverifiable constructively. In this essay we consider some arguments for, and consequences of, an interpretation of classical foundational concepts in which such truth is defined effectively.

5.30 Gödel's Incompleteness Theorems hold vacuously . . . *(July 2002)*

(This essay has been completely rewritten and superceded by a later essay)

5.31 Omega-inconsistency in Gödel's formal system: a constructive proof of the Entscheidungsproblem . . . *(June 2002)*

If we apply an extension of the Deduction meta-Theorem to Gödel's meta-reasoning of "undecidability", we can conclude that Gödel's formal system of Arithmetic is not omega-consistent. If we then interpret $[(\forall x)F(x)]$ as "There is a general, x-independent, routine to establish that $F(x)$ holds for all x", instead of as "$F(x)$ holds for all x", it follows that a constructively interpreted omega-inconsistent system proves Hilbert's Entscheidungsproblem negatively.

5.32 Reviewing Gödel's and Rosser's meta-reasoning of "undecidability" . . . *(April 2002)*

We review the classical conclusions drawn from Gödel's meta-reasoning establishing an undecidable proposition GUS in standard PA. We argue that, for any given set of numerical values of its free variables, every recursive arithmetical relation can be expressed formally in PA by different, but formally equivalent, propositions. We argue that this asymmetry yields alternative Representation and Self-reference meta-Lemmas. We argue that Gödel's meta-reasoning can thus be expressed avoiding any appeal to the truth of propositions in the standard interpretation IA of PA. We argue that this now establishes GUS as decidable, and PA as omega-inconsistent. We argue further that Rosser's extension of Gödel's meta-reasoning is invalid, and involves an intuitionistically objectionable deduction.

5.33 Paradox regained: Life beyond Gödel's shadow . . . *(March 2002)*

5.34 Beyond Gödel: Simply consistent constructive systems of first order Peano's Arithmetic that do not yield undecidable propositions by Gödel's reasoning . . . *(March 2002)*

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