

Gödel's Incompleteness Theorems hold vacuously

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Gödel's Theorem XI essentially states that, if there is a \mathbf{P} -formula $[\mathbf{Con}(\mathbf{P})]$ whose standard interpretation is equivalent to the assertion “ \mathbf{P} is consistent”, then $[\mathbf{Con}(\mathbf{P})]$ is not \mathbf{P} -provable. We argue that there is no such formula.

1.0 Introduction

Gödel's First Incompleteness Theorem

Theorem VI of Gödel's seminal 1931 paper [\[Go31a\]](#), commonly referred to as “Gödel's First Incompleteness Theorem”, essentially asserts:

Meta-theorem 1: Every omega-consistent formal system \mathbf{P} of Arithmetic contains a proposition “ $[(\mathbf{A}x)\mathbf{R}(x, p)]$ ” such that both “ $[(\mathbf{A}x)\mathbf{R}(x, p)]$ ” and “ $[\sim(\mathbf{A}x)\mathbf{R}(x, p)]$ ” are not \mathbf{P} -provable.

In an earlier essay [\[An02b\]](#), we argue, however, that a constructive interpretation of Gödel's reasoning establishes that any formal system of Arithmetic is omega-inconsistent.

It follows from this that Gödel's Theorem VI holds vacuously.

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Gödel's Second Incompleteness Theorem

In this essay, we now argue that Theorem XI of Gödel's paper [\[Go31a\]](#), commonly referred to as "Gödel's Second Incompleteness Theorem", also holds vacuously.

1.1 Notation

We generally follow the notation of Gödel [\[Go31a\]](#). However, we use the notation " $(\mathbf{A}x)$ ", whose standard interpretation is "for all x ", to denote Gödel's special symbolism for Generalisation.

We use square brackets to indicate that the expression (including square brackets) only denotes the string² named within the brackets. Thus, " $[(\mathbf{A}x)]$ " is not part of the formal system \mathbf{P} , and would be replaced by Gödel's special symbolism for Generalisation in order to obtain the actual string in which it occurs.

Following Gödel's definitions of well-formed formulas³, we note that juxtaposing the string " $[(\mathbf{A}x)]$ " and the formula⁴ " $[\mathbf{F}(x)]$ " is the formula " $[(\mathbf{A}x)\mathbf{F}(x)]$ ", juxtaposing the symbol " $[\sim]$ " and the formula " $[\mathbf{F}]$ " is the formula " $[\sim\mathbf{F}]$ ", and juxtaposing the symbol " $[\forall]$ " between the formulas " $[\mathbf{F}]$ " and " $[\mathbf{G}]$ " is the formula " $[\mathbf{F}\forall\mathbf{G}]$ ".

The numerical functions and relations in the following are defined explicitly by Gödel [\[Go31a\]](#). The formulas are defined implicitly by his reasoning.

² We define a "string" as any concatenation of a finite set of the primitive symbols of the formal system under consideration.

³ We note that all "well-formed formulas" of \mathbf{P} are "strings" of \mathbf{P} , but all "strings" of \mathbf{P} are not "well-formed formulas" of \mathbf{P} .

⁴ By "formula", we shall mean a "well-formed formula" as defined by Gödel.

1.2 Definitions

We take \mathbf{P} to be Gödel's formal system, and define ([\[Go31a\]](#), Theorem VI, p24-25):

- (i) " $\mathbf{Q}(x, y)$ " as Gödel's recursive numerical relation " $\sim x\mathbf{B}(\mathbf{Sb}(y \ 19|\mathbf{Z}(y)))$ ".
- (ii) " $[\mathbf{R}(x, y)]$ " as a formula that represents " $\mathbf{Q}(x, y)$ " in the formal system \mathbf{P} .

(The existence of such a formula follows from Gödel's Theorem VII (1).)
- (iii) " q " as the Gödel-number of the formula " $[\mathbf{R}(x, y)]$ " of \mathbf{P} .
- (iv) " p " as the Gödel-number of the formula " $[(\mathbf{A}x)][\mathbf{R}(x, y)]$ "⁵ of \mathbf{P} .
- (v) " $[p]$ " as the numeral that represents the natural number " p " in \mathbf{P} .
- (vi) " r " as the Gödel-number of the formula " $[\mathbf{R}(x, p)]$ " of \mathbf{P} .
- (vii) " $17\mathbf{Gen}r$ " as the Gödel-number of the formula " $[(\mathbf{A}x)][\mathbf{R}(x, p)]$ " of \mathbf{P} .
- (viii) " $\mathbf{Neg}(17\mathbf{Gen}r)$ " as the Gödel-number of the formula " $[\sim][(\mathbf{A}x)][\mathbf{R}(x, p)]$ " of \mathbf{P} .
- (ix) " $\mathbf{R}(x, y)$ " as the arithmetical interpretation⁶ of the formula " $[\mathbf{R}(x, y)]$ " of \mathbf{P} .

⁵ We note that " $[(\mathbf{A}x)][\mathbf{R}(x, y)]$ " and " $[(\mathbf{A}x)\mathbf{R}(x, y)]$ " denote the same formula of \mathbf{P} .

⁶ We define "interpretation" as in Mendelson ([\[Me64\]](#), p49). We note that the interpreted arithmetic expression " $\mathbf{F}(x)$ " is obtained from the formula $[\mathbf{F}(x)]$ by replacing every primitive, undefined symbol of \mathbf{PA} in the formula $[\mathbf{F}(x)]$ by an interpreted arithmetic symbol (i.e. a symbol that is a shorthand notation for some uniquely meaningful, intuitive, arithmetic concept).

So the \mathbf{PA} -formula $[(\mathbf{A}x)\mathbf{F}(x)]$ interprets as the arithmetic expression " $(\mathbf{A}x)\mathbf{F}(x)$ ", and the \mathbf{PA} -formula $[\sim(\mathbf{A}x)\mathbf{F}(x)]$ as the arithmetic expression " $\sim(\mathbf{A}x)\mathbf{F}(x)$ ".

(“ $\mathbf{R}(x, y)$ ” is defined by Gödel’s Theorem VII ([\[Go31a\]](#), p29), where it is proved equivalent to “ $\mathbf{Q}(x, y)$ ”.)

(x) " $\mathbf{Wid}(\mathbf{P})$ " as the number-theoretic assertion " $(\exists x)(\mathbf{Form}(x) \ \& \ \sim\mathbf{Bew}(x))$ ".

(" $\mathbf{Wid}(\mathbf{P})$ " is defined by Gödel ([\[Go31a\]](#), p36) as equivalent to the meta-assertion “ \mathbf{P} is consistent”.)

(xi) " $[\mathbf{Con}(\mathbf{P})]$ " as the formula that represents " $\mathbf{Wid}(\mathbf{P})$ " in the formal system \mathbf{P} .

(xii) " w " as the Gödel-number of the formula " $[\mathbf{Con}(\mathbf{P})]$ " of \mathbf{P} ([\[Go31a\]](#), p37).

(xiii) " $\mathbf{Con}(\mathbf{P})$ " as the arithmetic interpretation of the formula " $[\mathbf{Con}(\mathbf{P})]$ " of \mathbf{P} .

1.3 Gödel’s semantic theses in Theorem XI

We begin by noting some semantic theses that underlie Gödel’s proof of, and the conclusions he draws from, his Theorem XI ([\[Go31a\]](#), p36).

Thesis 1: If a formula $[\mathbf{F}]$ is \mathbf{P} -provable, then its standard interpretation “ \mathbf{F} ” is a true arithmetic assertion.

Thesis 2: “ \mathbf{P} is consistent”, abbreviated “ $\mathbf{Wid}(\mathbf{P})$ ”, can be meaningfully defined as equivalent to the number-theoretic proposition “ $(\exists x)(\mathbf{Form}(x) \ \& \ \sim\mathbf{Bew}(x))$ ” ([\[Go31a\]](#), p36, footnote 63).

Thesis 3: “ $\mathbf{Wid}(\mathbf{P})$ ” is equivalent to some arithmetic assertion " $\mathbf{Con}(\mathbf{P})$ ", that is the interpretation of a \mathbf{P} -formula " $[\mathbf{Con}(\mathbf{P})]$ ".

We also note that, classically, the equivalent meta-assertions “[$(\mathbf{Ax})\mathbf{F}(x)$] is a true proposition under the interpretation \mathbf{I} of \mathbf{PA} ”, and “ $(\mathbf{Ax})\mathbf{F}(x)$ is a true proposition of the interpretation \mathbf{I} of \mathbf{PA} ”, are merely shorthand notations for the meta-assertion “ $\mathbf{F}(x)$ is satisfied for any given value of x in the domain of the interpretation \mathbf{I} of \mathbf{PA} ” ([\[Me64\]](#), p51).

1.4 A meta-theorem

We now argue that Gödel's *Thesis 3* is false.

Meta-theorem 2: There is no **P**-formula that asserts, under interpretation, that **P** is consistent.

Proof: We assume that Gödel's *Thesis 3* is true, and there is some formula [**Con(P)**] of the formal system **P** such that, under the standard interpretation:

"**Con(P)**" is a true arithmetical relation \Leftrightarrow **P** is a consistent formal system.

We take this as equivalent to the assertion "[**Con(P)**] is a **P**-formula that asserts, under interpretation, that **P** is consistent".

- (i) By the definition of consistency⁷, [**Con(P)**] is **P**-provable if **P** is inconsistent - since every formula of an inconsistent **P** is a consequence of the Axioms and Rules of Inference of **P**⁸.
- (ii) Now, if [**Con(P)**] were **P**-provable then, by *Thesis 1*, we would conclude, under the standard interpretation, that:

"**Con(P)**" is a true arithmetical assertion.

We would further conclude, by *Theses 2 and 3*, the meta-assertion:

P is a consistent formal system.

However, this would be a false conclusion, since **P** may be inconsistent.

⁷ We take Mendelson's Corollary 1.15 ([Me64], p37), as the classical definition of "consistency".

⁸ This follows from ([Me64], p37, Corollary 1.15).

(iii) It follows that we cannot conclude from the \mathbf{P} -provability of $[\mathbf{Con}(\mathbf{P})]$ that \mathbf{P} is consistent. Hence we cannot have any formula $[\mathbf{Con}(\mathbf{P})]$ such that:

" $\mathbf{Con}(\mathbf{P})$ " is a true arithmetical assertion \implies " $\mathbf{Wid}(\mathbf{P})$ " is a true number-theoretic assertion.

(iv) We conclude that Gödel's *Thesis 3* is false, and there is no \mathbf{P} -formula $[\mathbf{Con}(\mathbf{P})]$ such that:

" $\mathbf{Con}(\mathbf{P})$ " is a true arithmetical relation \iff \mathbf{P} is a consistent formal system.

1.5 A meta-lemma

From the above we may also conclude that⁹:

Meta-lemma 1: Although a primitive recursive relation, and the interpretation of its formal representation, are always arithmetically equivalent, they are not always formally equivalent.

1.6 Gödel's Proof of Theorem XI

The question arises: Does *Meta-theorem 2* contradict Gödel's Theorem XI [\[Go31a\]](#)?

Now Gödel's Theorem XI [\[Go31a\]](#) is essentially the following assertion.

Meta-theorem 3: The consistency of \mathbf{P} is not provable in \mathbf{P} .

⁹ A possible cause, and significance, of such non-equivalence is discussed in ([\[An01\]](#), [Chapter 2](#)), and highlighted particularly in ([\[An01\]](#), [Chapter 2, Section 2.9](#)).

Proof: Gödel [\[Go31a\]](#) argues that:

- (i) If \mathbf{P} is assumed consistent, then the following number-theoretic assertions follow from his Theorems V, VI and his definition of “ $\mathbf{Wid}(\mathbf{P})$ ”.

$$\mathbf{Wid}(\mathbf{P}) \Rightarrow \sim \mathbf{Bew}(17\mathbf{Genr})$$

$$\mathbf{Wid}(\mathbf{P}) \Rightarrow (\mathbf{Ax})\sim x\mathbf{B}(17\mathbf{Genr})$$

$$17\mathbf{Genr} = \mathbf{Sb}(p \ 19|\mathbf{Z}(p))$$

$$\mathbf{Wid}(\mathbf{P}) \Rightarrow (\mathbf{Ax})\sim x\mathbf{B}(\mathbf{Sb}(p \ 19|\mathbf{Z}(p)))$$

$$\mathbf{Q}(x, y) \Leftrightarrow \sim x\mathbf{B}(\mathbf{Sb}(y \ 19|\mathbf{Z}(y)))$$

$$(x)\mathbf{Q}(x, y) \Leftrightarrow (\mathbf{Ax})\sim x\mathbf{B}(\mathbf{Sb}(y \ 19|\mathbf{Z}(y)))$$

$$\mathbf{Wid}(\mathbf{P}) \Rightarrow (\mathbf{Ax})\mathbf{Q}(x, p)$$

- (ii) Assuming that $[(\mathbf{Ax})\mathbf{R}(x, p)]$ asserts its own provability¹⁰, Gödel concludes from the above that the instantiation:

$$w\mathbf{Imp}(17\mathbf{Genr}),$$

of the recursive number-theoretic relation “ $x\mathbf{Imp}y$ ”, is a true numerical assertion.

- (iii) From this, he concludes that:

$$\text{“}[(\mathbf{Con}(\mathbf{P})) \Rightarrow [(\mathbf{Ax})\mathbf{R}(x, p)]]\text{” is } \mathbf{P}\text{-provable.}$$

¹⁰ We argue, in a companion essay, that this semantic assumption is a consequence of construing “ $(\mathbf{Ax})\mathbf{F}(x)$ ” non-constructively. It reflects the thesis that implicit Platonist assumptions underlie Gödel’s reasoning both in the proofs of, and the conclusions he draws from, various meta-propositions that he asserts as Theorems in [\[Go31a\]](#).

(iv) Now, in his Theorem VI, Gödel [Go31a] argues that, if \mathbf{P} is assumed consistent, then $[(\mathbf{A}\mathbf{x})\mathbf{R}(\mathbf{x}, \mathbf{p})]$ is not \mathbf{P} -provable. He thus concludes that, if \mathbf{P} is assumed consistent, then $[\mathbf{Con}(\mathbf{P})]$ too is not \mathbf{P} -provable.

(v) Implicitly assuming that *Thesis 3* is true, and so:

" $\mathbf{Con}(\mathbf{P})$ " is a true arithmetical assertion \iff " $\mathbf{Wid}(\mathbf{P})$ " is a true number-theoretic assertion,

Gödel further concludes that (iv) is equivalent to asserting that the consistency of \mathbf{P} is not provable in \mathbf{P} .

1.7 Conclusion

However, since, by *Meta-theorem 2*, Gödel's *Thesis 3* is an invalid implicit assumption, we conclude that Gödel's Theorem XI is essentially the vacuous meta-assertion:

Meta-theorem 4: If there is a \mathbf{P} -formula $[\mathbf{Con}(\mathbf{P})]$ whose standard interpretation is equivalent to the assertion " \mathbf{P} is consistent", then $[\mathbf{Con}(\mathbf{P})]$ is not \mathbf{P} -provable.

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