Why Hilbert's and Brouwer's interpretations of quantification are complementary and not contradictory

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01: Aristotle's particularisation

We consider the thesis that there is an implicit ambiguity in interpreting quantification, whose roots trace back to the non-finitary assumption of:

- ***An*** 'unspecified' element
- ***In*** a fundamental tenet of Aristotle's logic of predicates.

Namely, the semantic postulation that:

- ***If*** it is not the case that, for any specified x, $F(x)$ does not hold,
- ***Then*** there exists an unspecified x such that $F(x)$ holds.

Where 'holds' is to be understood semantically in Tarski's implicit sense:

- ***That*** 'Snow is white' holds as a true assertion if, and only if,
- ***It*** can be objectively determined, on the basis of evidence, that snow *is* white.

02: The significance of Hilbert's ε-calculus

Now, Hilbert defined a formal logic L_ε, in which he sought to capture the essence of:

- ***Aristotle's*** 'unspecified' x as $[\varepsilon_x(F(x))]$.

Hilbert showed:

- ***That*** the universal and existential quantifiers can be defined formally in L_ε in terms of his ε-operator as follows:
  - $[(\forall x)F(x) \leftrightarrow F(\varepsilon_x(\neg F(x)))]$
  - $[(\exists x)F(x) \leftrightarrow F(\varepsilon_x(F(x)))]$
- ***And*** that Aristotle's logic is a sound interpretation of the formal logic L_ε
  - ***If*** $[\varepsilon_x(F(x))]$ can be semantically interpreted as postulating the existence of some 'unspecified' x satisfying $F(x)$.

Definition

An interpretation (*model*) $\mathcal{I}$ of a formal language L, over a domain D, is sound relative to an assignment of truth values $T_{\mathcal{I}}$ to the formulas of L if, and only if, the axioms of L interpret as true, and the rules of inference of L preserve truth, over D under $\mathcal{I}$ relative to the assignment of truth values $T_{\mathcal{I}}$.

03: Hilbert's interpretation of quantification

Thus, Hilbert's interpretation of universal quantification—under any objective method T_H of assigning truth values to the sentences of a formal logic L—is that:

- ***The*** interpreted sentence $(\forall x)F(x)$ is defined as holding
  - ***If***, and only if, $F(a)$ holds whenever $\neg F(a)$ holds for some unspecified a;
  - ***Which*** implies that $\neg F(a)$ does not hold for any unspecified a if L is consistent,
- ***And*** so $(\forall x)F(x)$ holds,
  - ***If***, and only if, $F(a)$ holds for any unspecified a.

Whilst Hilbert's interpretation of existential quantification is the postulation that:

- ***The*** sentence $(\exists x)F(x)$ holds,
  - ***If***, and only if, $F(a)$ holds for some unspecified a.

04: Brouwer's objection

Brouwer's objection to such an 'unspecified' interpretation of quantification was that:

- ***For*** an interpretation to be considered sound relative to T_H,
- ***When*** the domain is infinite,
- ***Decidability***
    - ***Under*** the interpretation
    - ***Must*** be constructively verifiable
    - ***In*** some intuitive, and mathematically acceptable, sense of the term 'constructive'.

Two questions arise:

- Is Brouwer's objection relevant today?
- If so, can we interpret quantification 'constructively'?

06: The standard interpretation *M* of PA

The perspective we choose for addressing these issues is that of:

- ***The*** structure $\mathbb{N}$ of the natural numbers,

Which serves for a definition of today's:

- ***Standard*** *interpretation* ***M*** of the first-order Peano Arithmetic PA,
- ***Where*** we do not admit 'unspecified' natural numbers whilst defining quantification under ***M***.

However, we are then faced with the ambiguity:

07: Distinguishing between *For any* and *For all*

- ***Is*** the PA-formula $[(\forall x)F(x)]$[1]
  - ***To*** be interpreted constructively as: '*For any n,* $F(n)$',
  - ***Which*** holds if, and only if,
  - ***For*** *any specified* natural number n,
  - ***There*** is *algorithmic evidence* that $F(n)$ holds in $\mathbb{N}$?
- ***Or*** is $[(\forall x)F(x)]$
  - ***To*** be interpreted finitarily as: '*For all n,* $F(n)$',
  - ***Which*** holds if, and only if,
  - ***There*** is *algorithmic evidence* that,
  - ***For*** *any specified* natural number n, $F(n)$ holds in $\mathbb{N}$?

Where:

Definition

A natural number n is defined as specifiable in $\mathbb{N}$ if, and only if, it can be explicitly denoted as a PA-numeral by a PA-formula that interprets as an algorithmically computable[a] constant in $\mathbb{N}$.

[a] *As detailed in Definition 3.*

[1] *Square brackets identify and differentiate a formula from its interpretation.*

08: The standard interpretation of quantification in PA

Keeping this distinction in mind, we note that:

- ***If*** $F^*(x)$ denotes in $\mathbb{N}$ the relation that interprets the PA-formula $[F(x)]$ under todays standard interpretation ***M***,
- ***And***, if we assume that there is an objective method T_M of assigning truth values to the formulas of PA under ***M***,
- ***Then***, in the underlying first-order logic FOL of PA:
  - ***Which*** today favours evidence-based interpretation
  - ***Where*** we view the *values* of a simple functional language as specifying evidence for propositions in a constructive logic . . .

09: The standard interpretation of PA over $\mathbb{N}$

It would seem that:

- ***(1a)***: The formula $[(\forall x)F(x)]$ is ***defined*** as true in ***M***
  - ***Relative*** to the standard truth assignment T_M
  - ***If***, and only if, for any n, $F^*(n)$ holds in ***M***;
- ***(1b)***: The formula $[(\exists x)F(x)]$ is an abbreviation of $[\neg(\forall x)\neg F(x)]$,
  - ***And*** is ***defined*** as true in ***M*** relative to T_M
  - ***If***, and only if, it is not the case that, for any n, $\neg F^*(n)$ holds in ***M***;
- ***(1c)***: The sentence $F^*(n)$ is ***postulated*** as holding in ***M***
  - ***For*** some unspecified natural number n
  - ***If***, and only if, it is not the case that,
  - ***For*** any n, $\neg F^*(n)$ holds in ***M***.

If so, then (1a), (1b) and **(1c)** together interpret $[(\forall x)F(x)]$ and $[(\exists x)F(x)]$ under ***M*** as intended by Hilbert's ε-function, and attract Brouwer's objection.

This would, then, answer question (a).

10: The Law of the Excluded Middle and (1c)

Since definitions (1a) and (1b) are constructive:

- ***Our*** thesis is that the implicit target of Brouwer's objection is:
  - ***The*** semantic ***postulation*** **(1c)**[2],
  - ***Which*** appeals to Platonically non-constructive,
  - ***Rather*** than intuitively constructive, plausibility.

We note that this conclusion about Brouwer's essential objection:

- ***Apparently*** differs from conventional intuitionistic wisdom,
- ***Which*** would implicitly deny appeal to **(1c)**, in an interpretation of PA,
- ***By*** explicitly denying the FOL theorem [P v ¬P] (*Law of the Excluded Middle*);
  - ***Even*** though denying appeal to **(1c)** in an interpretation of PA
  - ***Does*** not entail denying the FOL theorem [P v ¬P].

[2] ***(1c)***: The sentence $F(n)$ is *implicitly* postulated as holding in ***M*** for some unspecified natural number n if, and only if, it is not the case that, for any specified natural number n, we may conclude on the basis of evidence-based reasoning that $F(n)$ does not hold in ***M***.

We therefore re-phrase question (b) more specifically:

- ***Can*** we define an interpretation of PA over $\mathbb{N}$ that does not appeal to the semantic ***postulation*** **(1c)**?
  - ***Where*** we do not ***postulate*** that the sentence $F(n)$ holds in ***M*** for some unspecified natural number n if, and only if, it is not the case that, for any specified n, $\neg F(n)$ holds in ***M***.

12: The interpretation *B* of PA over $\mathbb{N}$

Now, we can, indeed, define another interpretation ***B*** of PA over $\mathbb{N}$, under which:

- ***(2a)***: The formula $[(\forall x)F(x)]$ is ***defined*** as true in ***B***
  - ***Relative*** to a finitary truth assignment T_B
  - ***If***, and only if, for all n, $F^*(n)$ holds in ***B***;
- ***(2b)***: The formula $[(\exists x)F(x)]$ is an abbreviation of $[\neg(\forall x)\neg F(x)]$,
  - ***And*** is ***defined*** as true in ***B*** relative to T_B
  - ***If***, and only if, it is not the case that, for all n, $\neg F^*(n)$ holds in ***B***.

13: *B* is a finitary interpretation of PA

We show that[3] ***B*** is a finitary interpretation of PA,

- ***Since*** all the theorems of first-order PA interpret as finitarily true in ***B*** relative to T_B;
- ***From*** which we conclude finitarily that PA—and ipso facto FOL—are consistent,
  - ***So*** we need not deny the *Law of the Excluded Middle*
  - ***In*** order to ensure a finitary interpretation of quantification
  - ***Under*** an interpretation of PA.

This answers question (b).

[3] *As detailed in Theorem 8.*

14: The interpretations *M* and *B* are complementary

So, if we admit both the constructive and finitary interpretations of the PA-formula $[(\forall x)F(x)]$ as logically unobjectionable:

- ***Then*** the two interpretations ***M*** and ***B*** of PA over the structure $\mathbb{N}$
  - ***Can*** be viewed as complementary rather than contradictory.

15: Evidence-based reasoning

We note that the complementarity is rooted in Tarski's classic definitions:

- ***Which*** permit an intelligence,
  - ***Whether*** human
  - ***Or*** mechanistic,
- ***To*** admit,
  - ***Finitary***,
  - ***Evidence-based***,
  - ***Inductive***
- ***Definitions***
  - ***Of*** the satisfaction and truth
  - ***Of*** the **atomic** formulas of PA,
  - ***Over*** the domain N of the natural numbers,
- ***In*** two, essentially different, ways:

  (a) In terms of constructive algorithmic verifiabilty; and

  (b) In terms of finitary algorithmic computability.

What this means is that:

- ***If*** $[(\forall x)F(x)]$ is to be interpreted constructively as *'For any x, $F^*(x)$'*,
- ***Then*** it must be consistently read as:

Definition

A PA-formula $[F(x)]$ is algorithmically verifiable under an interpretation if, and only if, for any specified PA-term $[n]$, there is a deterministic algorithm[a] $AL_{(F,\ n)}$ which can provide objective evidence for deciding the truth or falsity of each formula in the finite sequence $[\{F(1), F(2), \ldots, F(n)\}]$ under the interpretation.

[a] A deterministic algorithm computes a mathematical function which has a unique value for any input in its domain, and the algorithm is a process that produces this particular value as output.

Whilst:

- ***If*** $[(\forall x)F(x)]$ is to be interpreted finitarily as *'For all x, $F^*(x)$'*,
- ***Then*** it must be consistently read as:

Definition

A PA-formula $[F(x)]$ is algorithmically computable under an interpretation if, and only if, there is a deterministic algorithm AL_F that can provide objective evidence for deciding the truth or falsity of each formula in the denumerable sequence $[\{F(1), F(2), \ldots\}]$ under the interpretation.

Now, although both definitions can be termed 'constructive':

- ***And*** every algorithmically computable number-theoretic relation is algorithmically verifiable,
  - ***The*** converse is false.[4]

Theorem

There are number-theoretical relations that are algorithmically verifiable but not algorithmically computable. □

An unintended significance of this is that the Church-Turing Thesis would not hold if we could define:

Definition

An arithmetical function is effectively computable if, and only if, it is algorithmically verifiable.

[4] *As detailed in Theorem 1.*

19: Decidability under Tarski's inductive definitions

Concerning the decidability of PA-formulas under Tarski's definitions, we note that[5]:

- ***If*** the atomic formulas of PA
- ***Interpret*** under an interpretation as decidable over the domain $\mathbb{N}$
- ***With*** respect to an objective assignment of truth values

  - ***Then*** the Π_n and Σ_n formulas of PA
  - ***Must*** also interpret as decidable over $\mathbb{N}$
  - ***With*** respect to the objective assignment of truth values.

5 *As detailed in the Satisfaction Theorem 2.*

Now it follows from the objective assignment T_M of algorithmically verifiable truth values under ***M*** that:[6]

Theorem

*The atomic formulas of PA are algorithmically verifiable as true or false under the standard interpretation **M**.*

From which we further conclude that:

Theorem

*The axioms of PA are algorithmically verifiable as true under the standard interpretation **M**, and the rules of inference of PA preserve the properties of algorithmically verifiable satisfaction and truth under **M**.* □

However, the interpretation ***M*** cannot claim to be finitary since:

- ***We*** cannot prove finitarily from Tarski's definitions and T_M whether, or not, a quantified PA formula $[(\forall x_i)R]$ is algorithmically verifiable as true under ***M***.

[6] *As detailed in Theorem 4 and Theorem 5.*

21: *M* proves PA consistent non-finitarily

We thus conclude that[7]:

Theorem

*If the PA-theorems interpret as algorithmically verifiable truths under the standard interpretation **M**[a], then PA is consistent.* □

[a] *As implied by Gerhard Gentzen's transfinite argument for the consistency of PA.*

- ***This*** suggests that the interpretation ***M*** of PA may be viewed as:
  - ***Circumscribing*** the ambit
  - ***Of*** non-finitary human reasoning,
  - ***About*** 'true' arithmetical propositions,
- ***If*** we see Aristotle's particularisation as:
  - ***A*** Platonic human-intelligence-specific inference,
  - ***That*** only a human-like intelligence can conceive of as holding,
  - ***Under*** the standard interpretation ***M*** of PA,
  - ***For*** deciding truth values in ***M*** under the assignment T_M.

[7] *As detailed in Theorem 6.*

Now it also follows from the objective assignment T_B of algorithmically computable truth values under ***B*** that:[8]

Lemma

*The atomic formulas of PA are algorithmically computable as true or as false under the interpretation **B**.*

From which we further conclude that:

Theorem

*The axioms of PA are algorithmically computable as true under the interpretation **B**, and the rules of inference of PA preserve the properties of algorithmically computable satisfaction and truth under **B**.* □

[8] *As detailed in Theorem 7 and Theorem 8.*

We then show that:[9]

Theorem

A PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$ . . . Provability Theorem for PA.

Since PA-provability is finitary, the assignment T_B of algorithmically computable truth values under the interpretation ***B*** is therefore finitarily decidable under Tarski's definitions.

Hence the PA-theorems interpret as finitary truths under ***B***, and we have a finitary proof, without appeal to Aristotle's particularisation (1c), that:

Theorem

PA is consistent. □

The finitary interpretation ***B*** may thus be viewed as:

- ***Circumscribing*** the ambit,
- ***Of*** finitary mechanistic reasoning
- ***About*** 'true' arithmetical propositions.

[9] *As detailed in Theorem 10 and Theorem 9.*

24: Gödel's arithmetical proposition $[(\forall x)R(x)]$

We finally consider the status of 'unspecified' natural numbers, and their putative representation as PA-terms (numerals) under a rule of deduction such as Rosser's Rule C, where we note that Gödel has defined:

- ***An*** arithmetical proposition $[(\forall x)R(x)]$ which is not PA-provable,
- ***Even*** though $[R(n)]$ is PA-provable for any specified PA-numeral $[n]$,

Now, we conclude from the Provability Theorem for PA that:[10]

Corollary

In any model of PA, Gödel's arithmetical formula $[R(x)]$ interprets as an algorithmically verifiable, but not algorithmically computable, relation over $\mathbb{N}$.

Corollary

The negation $[\neg(\forall x)R(x)]$ of Gödel's arithmetical proposition is provable in PA. □

Corollary

PA is not ω-consistent.

[10] *As detailed, contrary to accepted dogma, in Corollary 2, Corollary 3 and Corollary 4.*

25: PA can define only algorithmically computable natural numbers

So, since the negation $[\neg(\forall x)R(x)]$ of Gödel's arithmetical proposition is provable in PA, it admits the non-finitary conclusion:

- ***That*** there is an 'unspecified' *natural number* q,
  - ***For*** which the sentence $R^*(q)$ is false in $\mathbb{N}$ under ***M***,
  - ***Even*** though $[R(n)]$ is PA-provable for any specified numeral $[n]$;
- ***Which*** implies that the PA-numeral corresponding uniquely under a successor function to an unspecified natural number q:
  - ***Cannot*** be specified within any PA formula,
  - ***Even*** though q must lie in the domain N of the *natural numbers*
  - ***Which*** is defined completely by the semantics of Dedekind's second order Peano Postulates.
- ***This*** also means that we cannot use Rosser's Deduction Rule C within a PA-proof sequence, since it follows from the Provability Theorem for PA that:[11]

Theorem

A PA formula can only contain algorithmically computable terms.

[11] *As detailed in Theorem 11.*

26: Resolving the Poincaré-Hilbert debate

We conclude this overview by noting that:

- ***The*** complementarity suggested by the preceding perspective
- ***Can*** also be viewed as resolving the Poincaré-Hilbert debate in Hilbert's favour.

27: Interpretation *M* invalidates Poincaré's argument

Reason: Since the axioms of PA are algorithmically verifiable as true under the standard interpretation *M*[12],

- ***They*** invalidate Poincaré's argument, if we take this to mean that:

Poincaré

- ***The*** PA Axiom Schema of Finite Induction
- ***Cannot*** be justified
- ***Under*** the standard interpretation *M* of PA,
- ***As*** any such argument would necessarily
- ***Need*** to appeal to some form of infinite induction.

12 *As detailed in Theorem 5.*

28: Interpretation *B* validates Hilbert's belief

Whereas: Since the axioms of PA are algorithmically computable as true under the finitary interpretation ***B***.[13]

- ***They*** validate Hilbert's belief, if we take this to mean that:

Hilbert

- ***A*** finitary justification
- ***Of*** the PA Axiom Schema of Finite Induction
- ***Is*** possible
- ***Under*** some finitary interpretation of PA.

[13] *As detailed in Theorem 8.*

End

That concludes this overview of the arguments for

Why Hilbert's and Brouwer's interpretations of quantification

ought to be viewed

as complementary and not contradictory

Thank you