Research outline V

Can we *really* falsify truth by dictat?

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In a talk given on 25/5/96 at a BSPS conference in Oxford, on the Gödelian argument, J. R. Lucas (1996. *The Gödelian Argument: Turn Over the Page*) commented:

> ... in the case of First-order Peano Arithmetic there are Gödelian formulae (*many, in fact infinitely many, one for each system of coding*) which are not assigned truth-values by the rules of the system, and which could therefore be assigned either TRUE or FALSE, each such assignment yielding a logically possible, consistent system. These systems are random vaunts, all satisfying the core description of Peano Arithmetic.

Can we *really* falsify Arithmetical truth by such a dictat?

In other words, if $[(\forall x)R(x)]$ is the PA-unprovable Gödelian formula, which Gödel (1931. *On formally undecidable propositions of Principia Mathematica and related systems I.* In M. Davis. 1965. *The Undecidable*) interprets as Tarskian-true intuitively under the standard interpretation, can we *really* add its negation, $[\neg(\forall x)R(x)]$, as an axiom to PA, and *still* obtain a consistent system with a putative non-standard model of Arithmetic?

Here, $[(\forall x)R(x)]$ is the formula that Gödel (1931. p25) defines, and refers to, by its Gödel-number, $17Genr$.

Further, $[(\forall x)R(x)]$ is Tarskian-true under the standard interpretation if, and only if, the arithmetical relation, $R(x)$, holds for any given natural number.

Prima facie, Lucas's assumption appears to be based on a counter-intuitive interpretation, and extension, of Gödel's (1931. p27) assertion that, if an arithmetic such as PA is ω-consistent, then the system PA+$[\neg(\forall x)R(x)]$, say PA*, is consistent, but not ω-consistent.

Gödel defines PA as ω-consistent if, and only if, there is no PA-formula such as $[R(x)]$ for which:

(i) $[\neg(\forall x)R(x)]$ is PA-provable,

and:

(ii) $[R(n)]$ is PA-provable for any given numeral $[n]$ of PA.

Classically, any first order theory with equality is consistent if, and only if, it has a model. Gödel's postulation of the consistency of PA*, therefore, implies that it has a model.

Further, an implicit belief of classical theory is that any consistent first order mathematical theory can be interpreted *suitably* within a set theory such as Zermelo-Fraenkel (ZF), under which every model of ZF is a model of the theory.

Now, the *suitable* interpretation of the primitive symbols of PA in ZF which *transforms* the axioms of PA into theorems of ZF, whilst preserving its rules of inference, is the one (e. g. 1964. Elliott Mendelson. *Introduction to Mathematical Logic*. Van Norstrand. p192) that restricts the range of the interpreted variables to Cantor's first limit ordinal, ω, so that the PA-formula $[(\forall x)R(x)]$, for instance, *transforms* as $[(\forall x)((x \in \omega) \rightarrow R(x))]$.

Clearly, every model of ZF is, then, a model of the *transformed* axioms of PA.

However, does such an interpretation also assure us of a ZF model for the, similarly *transformed*, axioms of PA*?

Now, for PA* to have a model in ZF, in the above sense, we would need $[\neg(\forall x)((x \in \omega) \rightarrow R(x))]$ to be a theorem of ZF.

This, however, is not possible if ZF is consistent.

The reason: Since $[(\forall x)R(x)]$ is Tarskian-true under the standard model of PA, it follows, from Gödel's Completeness Theorem, that the ZF-formula $[(\forall x)((x \in \omega) \rightarrow R(x))]$ is already a theorem of ZF, as it is true in every model of ZF.

Gödel's Completeness Theorem: In any first-order predicate calculus, the theorems are precisely the logically valid well-formed formulas (*i. e. those that are true in every model of the calculus*).

The above argument holds for every interpretation of PA*, since (*see following section*) the Induction Axiom of PA would hold if, and only if—as in ZF—we can introduce an element, in the domain of the interpretation, which restricts the range of the interpreted variables to natural numbers.

Induction Axiom of PA: For any formula $F(x)$ of PA:
$F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(x')) \rightarrow (\forall x)F(x))$

So, was Gödel's postulation a case of a falsifiable conjecture, or are there alternative arguments for concluding that PA+$[\neg(\forall x)R(x)]$ has a non-standard model, and is, therefore consistent?

The following argument suggests that the question does not admit a simplistic answer.

DOES FIRST-ORDER PA *REALLY* ADMIT A NON-STANDARD MODEL?

Let $G(x)$ denote the PA-formula:

$[x = 0 \vee \neg(\forall y)\neg(x = y')]$

This translates, under every interpretation of PA, as:

Either x is 0, or x is a 'successor'.

Now, in every interpretation of PA:

(a) $G(0)$ is true;
(b) If $G(x)$ is true, then $G(x')$ is true.

By Gödel's completeness theorem:

(c) PA proves $[G(0)]$;
(d) PA proves $[G(x) \rightarrow G(x')]$.

By Generalisation:

(e) PA proves $[(\forall x)(G(x) \rightarrow G(x'))]$;

Generalisation in PA: $(\forall x)A$ follows from A.

By Induction:

(f) $[(\forall x)G(x)]$ is provable in PA.

Hence, except 0, every element in the domain of any interpretation of PA is a 'successor'.

Further, x can only be a 'successor' of a unique element in any interpretation of PA.

Now, since Cantor's first limit ordinal, ω, is not the 'successor' of any ordinal in the sense required by the PA axioms, and there are no infinitely descending sequences of ordinals, every set-theoretical interpretation of PA must, therefore, be restricted to the domain that consists only of the ordinal 0, and the ordinals that are the 'successors' of 0.

Although we *can* define a model of Arithmetic with an infinite descending sequence of elements (eg. Boolos, Burgess and Jeffrey. 2003. *Computability and Logic*. 4th ed. CUP. Section 25. 1, p303), any such model is isomorphic to the "*true arithmetic*" (ibid. p150. Ex. 12. 9) of the integers (*negative plus positive*), and *not* to any model of first-order PA (ibid. Corollary 25. 3, p306).

Moreover, since we cannot add a *non-successor* constant, say c, to PA such that $c \neq 0, c \neq 1, c \neq 2, \ldots$, we *cannot* apply the Compactness Theorem and the Löwenheim-Skolem Theorem (ibid. p306) to conclude that first-order PA has a non-standard model!

Hence PA admits no non-standard models!