An arithmetical perspective on Cantor’s Continuum Hypothesis

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Update of January 10, 2016

Abstract. The set-theoretical perspective on Cantor’s Continuum Hypothesis CH is well-known. We show that there is also an arithmetical perspective of CH that is based on first distinguishing between algorithmically verifiable functions and algorithmically computable functions. We then show how Gödel’s $\beta$-function uniquely corresponds each real number to an algorithmically verifiable arithmetical function that is representable in the first-order Peano Arithmetic PA. We conclude that the cardinality $2^{\aleph_0}$ of the real numbers cannot exceed the cardinality $\aleph_0$ of the integers. We further conclude that although the Continuum Hypothesis is independent of the axioms of the first order Set Theory ZF, CH follows from the axioms of the first order Peano Arithmetic PA.

Keywords. algorithmically computable, algorithmically verifiable, arithmetical, Cantor, Cohen, Continuum Hypothesis, finitary, first order, Gödel, Peano Arithmetic PA, set-theoretical, simple functional language, ZF.

2010 Mathematics Subject Classification. 03B10

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1. Introduction

The set-theoretical perspective on Cantor’s Continuum Hypothesis CH\(^1\) is well-known.

Kurt Gödel showed in 1939\(^2\) that CH is consistent with the usual Zermelo-Fraenkel (ZF) axioms for set theory if ZF is consistent. He defined a model of ZF in which both the Axiom of Choice (AC) and CH hold.

Paul Cohen showed in 1963\(^3\) that the negations of AC and CH are also consistent with ZF; in particular, CH can fail while AC holds in a model of ZF.

We shall argue that there is also an arithmetical perspective of CH that is based on first distinguishing\(^4\) between algorithmically verifiable number-theoretic functions and algorithmically computable number-theoretic functions.

We shall then show how Gödel’s $\beta$-function uniquely corresponds each real number to an algorithmically verifiable arithmetical function.

\(^1\)There is no set whose cardinality is strictly between the cardinality $\aleph_0$ of the integers and the cardinality $2^{\aleph_0}$ of the real numbers.

\(^2\)[Go40].

\(^3\)[Co66].

\(^4\)The distinction was introduced—and its significance highlighted—in [An12]. Since set-theoretic functions are defined extensionally, it is not obvious how—or even whether—this distinction can be reflected within ZF.
1.A. Algorithmically verifiable functions and algorithmically computable functions

Definition 1. A number-theoretical function $F(x)$ is algorithmically verifiable if, and only if, for any given natural number $n$, there is an algorithm $AL(F, n)$ which can provide objective evidence\(^5\) for deciding the value of each formula in the finite sequence $\{F(1), F(2), \ldots, F(n)\}$.

Definition 2. A number theoretical function $F(x)$ is algorithmically computable if, and only if, there is an algorithm $AL_F$ that can provide objective evidence for deciding the value of each formula in the denumerable sequence $\{F(1), F(2), \ldots\}$.

Although we shall not appeal to the following in this investigation, we note in passing that:

Lemma 1.1. There are algorithmically verifiable number theoretical functions which are not algorithmically computable.\(^8\)

Proof: Let $r(n)$ denote the $n^{th}$ digit in the decimal expansion $\sum_{n=1}^{\infty} r(n) \cdot 10^{-n}$ of a putatively given real number $\mathbb{R}$ in the interval $0 < \mathbb{R} \leq 1$. By the definition of a real number as the limit of a Cauchy sequence of rationals, it follows that $r(n)$ is an algorithmically verifiable number-theoretic function. Since every algorithmically computable real is countable\(^9\), Cantor's diagonal argument\(^10\) shows that there are real numbers that are not algorithmically computable. The Lemma follows. \(\square\)

1.B. Gödel’s $\beta$-function

We note that Gödel’s $\beta$-function is defined as\(^11\):

\[
\beta(x_1, x_2, x_3) = r m(1 + (x_3 + 1) \star x_2, x_1)
\]

\(^{5}\)“It is by now folklore ... that one can view the values of a simple functional language as specifying evidence for propositions in a constructive logic ...”; [Mu91].

\(^{6}\)We show in [An15] (§4, p.6) that the concept is well-defined; in the sense that the ‘algorithmic verifiability’ of the formulas of a formal language which contain logical constants can be inductively defined under an interpretation in terms of the ‘algorithmic verifiability’ of the interpretations of the atomic formulas of the language; further, that the formulas of the first order Peano Arithmetic PA are decidable under the standard interpretation of PA over the domain $\mathbb{N}$ of the natural numbers if, and only if, they are algorithmically verifiable under the interpretation. Moreover, the PA axioms are algorithmically verifiable as true over $\mathbb{N}$, and the PA rules of inference preserve algorithmically verifiable truth over the standard interpretation of PA. However, the compound formulas of PA cannot always be finitarily assigned algorithmically verifiable truth values under the standard interpretation of PA.

\(^{7}\)We show in [An15] (§4, p.6) that this concept too is well-defined; in the sense that the ‘algorithmic computability’ of the formulas of a formal language which contain logical constants can also be inductively defined under an interpretation in terms of the ‘algorithmic computability’ of the interpretations of the atomic formulas of the language; further, that the PA-formulas are decidable under a finitary interpretation of PA over $\mathbb{N}$ if, and only if, they are algorithmically computable under the interpretation. Moreover, the PA axioms are algorithmically computable as true over $\mathbb{N}$, and the PA rules of inference preserve algorithmically computable truth over the finitary interpretation of PA. In this case, though, the compound formulas of PA can be always be finitarily assigned algorithmically computable truth values under the finitary interpretation of PA.

\(^{8}\)We note that algorithmic computability implies the existence of an algorithm that can decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions, whereas algorithmic verifiability does not imply the existence of an algorithm that can decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions. From the point of view of a finitary mathematical philosophy, the significant difference between the two concepts could be expressed ([An15a], Thesis 1, p.11) by saying that we may treat the decimal representation of a real number as corresponding to a physically measurable limit—and not only to a mathematically definable limit—if and only if such representation is definable by an algorithmically computable function.

\(^{9}\)[Tu36].

\(^{10}\)[Kl52], pp.6-8.

\(^{11}\)[Me64], p.131.
where \( rm(x_1, x_2) \) denotes the remainder obtained on dividing \( x_2 \) by \( x_1 \).

We also note that:

**Lemma 1.2.** For any non-terminating sequence of values \( f(0), f(1), \ldots \), we can construct natural numbers \( b_k, c_k \) such that:

(i) \( j_k = \max(k, f(0), f(1), \ldots, f(k)) \);
(ii) \( c_k = j_k! \);
(iii) \( \beta(b_k, c_k, i) = f(i) \) for \( 0 \leq i \leq k \).

**Proof** This is a standard result\(^{12}\). □

Now we have the standard definition\(^{13}\):

**Definition 3.** A number-theoretic function \( f(x_1, \ldots, x_n) \) is said to be representable in the first order Peano Arithmetic \( PA \) if, and only if, there is a \( PA \) formula \([F(x_1, \ldots, x_{n+1})]\)\(^{14}\) with the free variables \([x_1, \ldots, x_{n+1}]\), such that, for any given natural numbers \( k_1, \ldots, k_{n+1} \):

(i) if \( f(k_1, \ldots, k_n) = k_{n+1} \) then \( PA \) proves: \([F(k_1, \ldots, k_n, k_{n+1})]\);
(ii) \( PA \) proves: \([\exists_1 x_{n+1})F(k_1, \ldots, k_n, x_{n+1})]\).

The function \( f(x_1, \ldots, x_n) \) is said to be strongly representable in \( PA \) if we further have that:

(iii) \( PA \) proves: \([\exists_1 x_{n+1})F(x_1, \ldots, x_n, x_{n+1})]\)

We also have that:

**Lemma 1.3.** \( \beta(x_1, x_2, x_3) \) is strongly represented in \( PA \) by \([Bt(x_1, x_2, x_3, x_4)]\), which is defined as follows:

\[
((\exists w)(x_1 = ((1 + (x_3 + 1) \ast x_2) \ast w + x_4) \land (x_4 < 1 + (x_3 + 1) \ast x_2))).
\]

**Proof** This is a standard result\(^{15}\). □

### 2. An arithmetical perspective on Cantor’s Continuum Hypothesis

**Theorem 2.1.** The cardinality \( 2^\aleph_0 \) of the real numbers cannot exceed the cardinality \( \aleph_0 \) of the integers.

**Proof:** Let \( \{r(n)\} \) be the denumerable sequence defined by the denumerable sequence of digits in the decimal expansion \( \sum_{n=1}^{\infty} r(n).10^{-n} \) of a putatively given real number \( R \) in the interval \( 0 < R \leq 1 \).

By lemma 1.2, for any given natural number \( k \), we can define natural numbers \( b_k, c_k \) such that, for any \( 1 \leq n \leq k \):

\[
\beta(b_k, c_k, n) = r(n).
\]

---

\(^{12}\) Me64, p.131, Proposition 3.22.

\(^{13}\) Me64, p.118.

\(^{14}\) We use square brackets simply as a convenience in informal argumentation to differentiate between a formal expression and its interpretation over some domain when there is a possibility of confusion between the two.

\(^{15}\) cf. Me64, p.131, proposition 3.21.
By lemma 1.3, \( \beta(b_k, c_k, n) \) is uniquely represented in the first order Peano Arithmetic PA by \([Bt(b_k, c_k, n, x)]\) such that, for any \(1 \leq n \leq k\):

If \( \beta(b_k, c_k, n) = r(n) \) then PA proves \([Bt(b_k, c_k, n, r(n))]\).

We now define the arithmetical formula \([R(b_k, c_k, n)]\) for any \(1 \leq n \leq k\) by:

\[ [R(b_k, c_k, n) = r(n)] \text{ if, and only if, PA proves } [Bt(b_k, c_k, n, r(n))]. \]

Hence every putatively given real number \( R \) in the interval \( 0 < R \leq 1 \) can be uniquely corresponded to an algorithmically verifiable arithmetical formula \([R(x)]\) since:

For any \( k \), the primitive recursivity of \( \beta(b_k, c_k, n) \) yields an algorithm \( AL(\beta, R, k) \) that provides objective evidence for deciding the unique value of each formula in the finite sequence \([R(1), R(2), \ldots, R(k)]\) by evidencing the truth under a sound interpretation of PA for:

\[
\begin{align*}
[R(1) &= R(b_k, c_k, 1)] \\
[R(b_k, c_k, 1) &= r(1)] \\
[R(2) &= R(b_k, c_k, 2)] \\
[R(b_k, c_k, 2) &= r(2)] \\
&\ldots \\
[R(k) &= R(b_k, c_k, k)] \\
[R(b_k, c_k, k) &= r(k)].
\end{align*}
\]

The correspondence is unique because, if \( R \) and \( S \) are two different putatively given reals in the interval \( 0 < R, S \leq 1 \), then there is always some \( m \) for which:

\[ r(m) \neq s(m). \]

Hence we can always find corresponding arithmetical functions \([R(n)]\) and \([S(n)]\) such that:

\[
\begin{align*}
[R(n) &= r(n)] \text{ for all } 1 \leq n \leq m. \\
[S(n) &= s(n)] \text{ for all } 1 \leq n \leq m. \\
[R(m) &\neq S(m)].
\end{align*}
\]

Since PA is first order, the cardinality of the reals cannot, therefore, exceed that of the integers.

The theorem follows. \( \Box \)

**Corollary 2.2.** \( \aleph_0 \leftrightarrow 2^{\aleph_0} \)

We also conclude that, although the Continuum Hypothesis is independent of the axioms of ZF if ZF is consistent, CH follows from the axioms of PA—which is finitarily provable as consistent\(^{16}\).

**References**


\(^{16}\)See [An12] and [An15].


