# A Post-Computationalist Evidence-Based Arithmetical Perspective on the Forcing of Non-Standard Models onto PA

**Bhupinder Singh Anand**

Update of January 1, 2016

**Abstract.** We show why some standard arguments for the existence of non-standard models of the first-order Peano Arithmetic PA ought to be considered foundationally fragile from a post-computationalist evidence-based arithmetical perspective within classical logic, rather than accepted as foundationally sound relative to an ante-computationalist perspective of set theory.

**Keywords.** Compactness, consistency, evidence-based reasoning, first-order, Gödel, natural numbers, non-standard model, $\omega$-consistency, Peano Arithmetic PA, standard interpretation, ZF.

**2010 Mathematics Subject Classification.** 03B10

## Contents



## 1. Introduction

Once we accept as logically sound the set-theoretically based meta-argument[1] that the first-order Peano Arithmetic PA[2] can be forced—by an ante-computat- ionalist interpretation of the Compactness Theorem—into admitting non-standard models which contain an 'infinite' integer, then the

[1]By which we mean arguments such as in [Ka91] (see pg.1), where the meta-theory is taken to be a set-theory such as ZF or ZFC, and the logical consistency of the meta-theory is not considered relevant to the argumentation.

[2]For purposes of this investigation we may take this to be a first order theory such as the theory S defined in [Me64], pp.102-103.

set-theoretical properties[3] of the algebraic and arithmetical structures of such putative models should perhaps follow without serious foundational reservation.

> *Compactness Theorem*: If every finite subset of a set of sentences has a model, then the whole set has a model[4].

We note that, even from a post-computationalist evidence-based arithmetical perspective[5] anchored *strictly within* the framework of classical logic[6], we can conclude incontrovertibly by the Compactness Theorem that if $Th(\mathbb{N})$ is the $\mathcal{L}_A$-theory of the standard model of Arithmetic (i.e., $Th(\mathbb{N})$ is the collection of all true $\mathcal{L}_A$-sentences)[7], then we may consistently add to it the following as an additional—not necessarily independent—axiom:

$(\exists y)(y > x)$.

However, we shall argue that even though $(\exists y)(y > x)$ is algorithmically computable (Definition 2) as always true in the standard model—whence all of its instances are in $Th(\mathbb{N})$—we cannot conclude[8] by the Compactness Theorem that $\cup_{k\in\mathbb{N}}\{Th(\mathbb{N}) \cup \{c > \underline{n} \mid n < k\}\}$ is consistent and has a model $M_c$ which contains an 'infinite' integer.

*Reason*: We shall argue that the condition '$k \in \mathbb{N}$' in the above definition of '$\cup_{k\in\mathbb{N}}\{Th(\mathbb{N}) \cup \{c > \underline{n} \mid n < k\}\}$' requires, first of all, that we must be able to extend $Th(\mathbb{N})$ by the addition of a 'relativised' axiom[9], such as:

$(\exists y)((x \in \mathbb{N}) \rightarrow (y > x))$.

Only then may we conclude that if a model $M_c$ of $\{Th(\mathbb{N}) \cup (\exists y)((x \in \mathbb{N}) \rightarrow (y > x))\}$ exists, then it must have an 'infinite' integer $c$ such that:

$M_c \models c > \underline{n}$

for all $n \in \mathbb{N}$.

However, we shall then argue that even this would not yield a model for $Th(\mathbb{N})$, since every model of $Th(\mathbb{N})$ is by definition a model of (the provable formulas of) PA, and we shall show that we cannot introduce a 'completed' infinity such as $\mathbb{N}$ into either PA or any model of PA!

## 1.A. A post-computationalist doctrine

More generally we shall argue that, if our interest is in the arithmetical properties of models of PA, then we first need to make explicit any appeal to non-constructive considerations such as Aristotle's particularisation (Definition 3).

We shall then argue that, even from a classical perspective, there are serious foundational, post-computationalist, reservations to accepting that a consistent PA can be forced by the Compactness Theorem into admitting non-standard models which contain elements other than the natural numbers.

*Reason*: Any arithmetical application of the Compactness Theorem to PA can neither ignore currently accepted post-computationalist doctrines of objectivity—nor contradict the evidence-based

[3] eg. [Ka91]; [Bo00]; [BBJ03], ch.25, p.302; [Ko06]; [Ka11].
[4] [BBJ03]. p.147.
[5] As introduced in [An12].
[6] By 'classical logic' we mean the standard first-order predicate calculus FOL where the Law of the Excluded Middle is a theorem, but we do not assume that FOL is $\omega$-consistent; i.e., we do not assume that Aristotle's particularisation (Definition 3) must hold under any interpretation of the logic.
[7] [Ka91], p.10-11.
[8] As argued in [Ka91], p.10-11.
[9] cf. [Fe92]; [Me64], p.192.

assignments of satisfaction and truth to the atomic formulas of PA (therefore to the compound formulas under Tarski's inductive definitions) in terms of either algorithmical verifiability or algorithmic computability[10]—as expressed, for instance, by the following:

> **Post-computationalist doctrine**
>
> "It is by now folklore ... that one can view the *values* of a simple functional language as specifying *evidence* for propositions in a constructive logic ..."[11].

The significance of this doctrine[12] is that it helps highlight how the algorithmically verifiable (Definition 1) formulas of PA define the classical non-finitary standard interpretation of PA over $N$[13] (to which standard arguments for the existence of non-standard models of PA critically appeal).

Accordingly, we shall show that standard arguments which appeal to the ante-computationalist interpretation of the Compactness Theorem—for forcing non-standard models of PA[14] which contain an 'infinite' integer—cannot admit constructive assignments of satisfaction and truth[15] (in terms of algorithmical verifiability) to the atomic formulas of their putative extension of PA.

We shall conclude that such arguments therefore questionably postulate by axiomatic fiat that which they seek to 'prove'!

## 1.B. Standard arguments for non-standard models of PA

In this limited investigation we shall consider only the following three standard arguments for the existence of non-standard models of the first-order Peano Arithmetic PA:

> $(i)$ If PA is consistent, then we obtain a non-standard model for PA which contains an 'infinite' integer by applying the Compactness Theorem to the union of the set of formulas that are satisfied or true in the classical 'standard' model of PA[16] and the countable set of all PA-formulas of the form $[c_n = S(c_{n+1})]$.
>
> $(ii)$ If PA is consistent, then we obtain a non-standard model for PA which contains an 'infinite' integer by adding a constant $c$ to the language of PA and applying the Compactness Theorem to the theory $\mathbf{P} \cup \{c > \underline{n} : \underline{n} = \underline{0}, \underline{1}, \underline{2}, \ldots\}$.
>
> $(iii)$ If PA is consistent, then we obtain a non-standard model for PA which contains an 'infinite' integer by adding the PA formula $[\neg(\forall x)R(x)]$ as an axiom to PA, where $[(\forall x)R(x)]$ is a Gödelian formula[17] that is unprovable in PA, even though $[R(n)]$ is provable in PA for any given PA numeral $[n]$[18].

---

[10] As introduced in [An12], §3.

[11] cf. [Mu91].

[12] Some of the—hitherto unsuspected—consequences of this doctrine are detailed in [An12].

[13] [An12], Corollary 2; 'non-finitary' because even though the Axiom Schema of Finite Induction interprets as true under the standard interpretation of PA over $N$ with respect to 'truth' as defined by the algorithmically verifiable formulas of PA, the compound formulas of PA are not decidable finitarily under the standard interpretation of PA over $N$ with respect to algorithmically verifiable 'satisfaction' and 'truth'.

[14] eg., [BBJ03], p.155, Lemma 13.3 (Model existence lemma).

[15] cf. The standard non-constructive set-theoretical assignment-by-postulation **(S5)** of the satisfaction properties **(S1)** to **(S8)** in [BBJ03], p.153, Lemma 13.1 (Satisfaction properties lemma), which appeals critically to Aristotle's particularisation.

[16] For purposes of this investigation we may take this to be an interpretation of PA as defined in [Me64], p.107.

[17] In his seminal 1931 paper [Go31], Kurt Gödel defines, and refers to, the formula corresponding to $[R(x)]$ only by its 'Gödel' number $r$ (op. cit., p.25, Eqn.(12)), and to the formula corresponding to $[(\forall x)R(x)]$ only by its 'Gödel' number 17 *Gen* $r$.

[18] [Go31], p.25(1).

We shall first argue that $(i)$ and $(ii)$—which appeal to Thoraf Skolem's ante-computationalist reasoning[19] for the existence of a non-standard model of PA—should be treated as foundationally fragile from a finitary, post-computationalist perspective within classical logic[20].

We shall then argue that although $(iii)$—which appeals to Kurt Gödel's (also ante-computationalist) reasoning[21] for the existence of a non-standard model of PA—does yield a model other than the classical 'standard' model of PA, we cannot conclude by even classical (albeit post-computationalist) reasoning that the domain is other than the domain $N$ of the natural numbers unless we make the non-constructive—and logically fragile—extraneous assumption that a consistent PA is necessarily $\omega$-consistent.

> ($\omega$-*consistency*): A formal system S is $\omega$-consistent if, and only if, there is no S-formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S-provable and, second, $[F(a)]$ is S-provable for any given S-term $[a]$.

## 2. Algorithmically verifiable formulas and algorithmically computable formulas

We begin by distinguishing between:

**Definition 1.** *An atomic formula $[F(x)]$[22] of PA is algorithmically verifiable under an interpretation if, and only if, for any given numeral $[n]$, there is an algorithm $AL_{(F,\ n)}$ which can provide objective evidence[23] for deciding the truth value of each formula in the finite sequence of PA formulas $\{[F(1)], [F(2)], \ldots, [F(n)]\}$ under the interpretation.*

The concept is well-defined in the sense that the 'algorithmic verifiability' of the formulas of a formal language which contain logical constants can be—albeit non-finitarily—defined under an interpretation in terms of the 'algorithmic verifiability' of the interpretations of the atomic formulas of the language[24].

However it can be shown that[25], under such an interpretation, the PA axioms are algorithmically verifiable as always true over $N$, and that the PA rules of inference preserve algorithmically verifiable truth over $N$.

We note further that the formulas of the first order Peano Arithmetic PA are decidable under the standard interpretation of PA over the domain $\mathbb{N}$ of the natural numbers if, and only if, they are algorithmically verifiable under the interpretation.

**Definition 2.** *An atomic formula $[F(x)]$ of PA is algorithmically computable under an interpretation if, and only if, there is an algorithm $AL_F$ that can provide objective evidence for deciding the truth value of each formula in the denumerable sequence of PA formulas $\{[F(1)], [F(2)], \ldots\}$ under the interpretation.*

[19] In [Sk34].

[20] By 'classical logic' we mean the standard first-order predicate calculus FOL where we neither deny the Law of the Excludeds Middle, nor assume that the FOL is $\omega$-consistent (i.e., we do not assume that Aristotle's particularisation must hold under any interpretation of the logic).

[21] In [Go31].

[22] *Notation*: For the sake of convenience, we shall use square brackets to indicate that the expression enclosed by them is to be treated as denoting a formula of a formal theory, and not as denoting an interpretation.

[23] [Mu91].

[24] [An12].

[25] A straightforward consequence of the evidence-based reasoning in [An12], by arguments paralleling those of Theorem 4.

This concept too is well-defined in the sense that the 'algorithmic computability' of the formulas of a formal language which contain logical constants can be—in this case finitarily—defined under an interpretation in terms of the 'algorithmic computability' of the interpretations of the atomic formulas of the language[26].

Moreover, it can be now shown that[27], under such an interpretation, the PA axioms are algorithmically computable as always true over $N$, and that the PA rules of inference preserve algorithmically computable truth over $N$.

Although we shall not appeal to the following in this paper, we note in passing that the foundational significance of the distinction between algorithmic verifiability and algorithmic computability for any theory of the real numbers (and of their extension) that has classically sought to be built upon the foundations of an arithmetic such as the first-order arithmetic PA[28] lies in the argument that:

**Lemma 2.1.** *There are algorithmically verifiable number theoretical functions which are not algorithmically computable.*[29]

**Proof**: If we accept as classically incontrovertible the definition of a real number $\mathbb{R}$ in the interval $0 < \mathbb{R} \leq 1$ as the limit $Lt_{n\to\infty} \sum_{i=1}^{n} r(i).2^{-i}$ of the Cauchy sequence $\{\sum_{i=1}^{n} r(i).2^{-i}\}$ of rationals, then $r(n)$ is an algorithmically verifiable number-theoretic function. Since every algorithmically computable real is countable[30], Cantor's diagonal argument[31] and Turing's Halting argument[32] together show that there are real numbers that are algorithmically verifiable but not algorithmically computable. The Lemma follows. □

## 3. Making non-finitary assumptions explicit

We next make explicit—and briefly review—a tacitly held fundamental tenet of classical logic which is unrestrictedly adopted as intuitively obvious by standard literature[33] that seeks to build upon the formal first-order predicate calculus FOL:

**Definition 3.** ***Aristotle's particularisation*** *This holds that from an assertion such as:*

> *'It is not the case that: For any given $x$, $P^*(x)$ does not hold',*

*usually denoted symbolically by '$\neg(\forall x)\neg P^*(x)$', we may always validly infer in the classical, Aristotlean, logic of predicates*[34] *that:*

---

[26] [An12].

[27] Theorem 4 in [An12].

[28] Such as, for instance, in Hardy's classic text [Ha60]; see also Edmund Landau's slim, but as charming as it is classically rigorous, text [La29].

[29] We note that algorithmic computability implies the existence of an algorithm that can decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions, whereas algorithmic verifiability does not imply the existence of an algorithm that can decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions. From the point of view of a finitary mathematical philosophy, the significant difference between the two concepts could be expressed ([An13]) by saying that we may treat the decimal representation of a real number as corresponding to a physically measurable limit—and not only to a mathematically definable limit—if and only if such representation is definable by an algorithmically computable function.

[30] [Tu36].

[31] [Kl52], pp.6-8.

[32] [Tu36].

[33] See [Hi25], p.382; [HA28], p.48; [Sk28], p.515; [Go31], p.32.; [Kl52], p.169; [Ro53], p.90; [BF58], p.46; [Be59], pp.178 & 218; [Su60], p.3; [Wa63], p.314-315; [Qu63], pp.12-13; [Kn63], p.60; [Co66], p.4; [Me64], pp.45, 47, 52(ii), 214(fn); [Nv64], p.92; [Li64], p.33; [Sh67], p.13; [Da82], p.xxv; [Rg87], p.xvii; [EC89], p.174; [Mu91]; [Sm92], p.18, Ex.3; [BBJ03], p.102.

[34] [HA28], pp.58-59.

*'There exists an unspecified $x$ such that $P^*(x)$ holds',*

*usually denoted symbolically by '$(\exists x)P^*(x)$'.*

### 3.A. The significance of Aristotle's particularisation for the first-order predicate calculus

Now we note that in a formal language the formula '$[(\exists x)P(x)]$' is an abbreviation for the formula '$[\neg(\forall x)\neg P(x)]$'; and that the commonly accepted interpretation of this formula tacitly appeals to Aristotlean particularisation.

However, as L. E. J. Brouwer had noted in his seminal 1908 paper on the unreliability of logical principles[35], the commonly accepted interpretation of this formula is ambiguous if interpretation is intended over an infinite domain.

Brouwer essentially argued that, even supposing the formula '$[P(x)]$' of a formal Arithmetical language interprets as an arithmetical relation denoted by '$P^*(x)$', and the formula '$[\neg(\forall x)\neg P(x)]$' as the arithmetical proposition denoted by '$\neg(\forall x)\neg P^*(x)$', the formula '$[(\exists x)P(x)]$' need not interpret as the arithmetical proposition denoted by the usual abbreviation '$(\exists x)P^*(x)$'; and that such postulation is invalid as a general logical principle in the absence of a means for constructing some putative object $a$ for which the proposition $P^*(a)$ holds in the domain of the interpretation.

Hence we shall follow the convention that the assumption that '$(\exists x)P^*(x)$' is the intended interpretation of the formula '$[(\exists x)P(x)]$'—which is essentially the assumption that Aristotle's particularisation holds over the domain of the interpretation—must always be explicit.

### 3.B. The significance of Aristotle's particularisation for PA

In order to avoid intuitionistic objections to his reasoning, Kurt Gödel introduced the syntactic property of $\omega$-consistency[36] as an explicit assumption in his formal reasoning in his seminal 1931 paper on formally undecidable arithmetical propositions[37].

Gödel explained at some length[38] that his reasons for introducing $\omega$-consistency explicitly was to avoid appealing to the semantic concept of classical arithmetical truth in Aristotle's logic of predicates (which presumes Aristotle's particularisation).

The two concepts are meta-mathematically equivalent in the sense that, if PA is consistent, then PA is $\omega$-consistent if, and only if, Aristotle's particularisation holds under the standard interpretation of PA[39].

## 4. The ambiguity in admitting an 'infinite' constant

We begin our consideration of standard arguments for the existence of non-standard models of PA which contain an 'infinite' integer by first highlighting and eliminating an ambiguity in the argument as it is usually found in standard texts[40]:

> "**Corollary**. There is a non-standard model of **P** with domain the natural numbers in which the denotation of every nonlogical symbol is an arithmetical relation or function.

[35] [Br08].
[36] The significance of $\omega$-consistency for the formal system PA is highlighted in[An12].
[37] [Go31], p.23 and p.28.
[38] In his introduction on p.9 of [Go31].
[39] For details see [An12].
[40] cf. [HP98], p.13, §0.29; [Me64], p.112, Ex. 2.

*Proof.* As in the proof of the existence of nonstandard models of arithmetic, add a constant $\infty$ to the language of arithmetic and apply the Compactness Theorem to the theory

$$\mathbf{P} \cup \{\infty \neq \mathbf{n}\colon n = 0, 1, 2, \ldots\}$$

to conclude that it has a model (necessarily infinite, since all models of **P** are). The denotations of $\infty$ in any such model will be a non-standard element, guaranteeing that the model is non-standard. Then apply the arithmetical Löwenheim-Skolem theorem to conclude that the model may be taken to have domain the natural numbers, and the denotations of all nonlogical symbols arithmetical."

... [BBJ03], p.306, Corollary 25.3.

## 4.A. We cannot force PA to admit a transfinite ordinal

The ambiguity lies in a possible interpretation of the symbol $\infty$ as a 'completed' infinity (such as Cantor's first transfinite ordinal $\omega$) in the context of non-standard models of PA. To eliminate this possibility we establish trivially that, and briefly examine why:

**Theorem 4.1.** *No model of PA can admit a transfinite ordinal under the standard interpretation of the classical logic FOL*[41].

*Proof* Let $[G(x)]$ denote the PA-formula:

$$[x = 0 \vee \neg(\forall y)\neg(x = Sy)]$$

Since Aristotle's particularisation is tacitly assumed under the standard interpretation of FOL, this translates in every model of PA, as:

If $x$ denotes an element in the domain of a model of PA, then either $x$ is 0, or $x$ is a 'successor'.

Further, in every model of PA, if $G(x)$ denotes the interpretation of $[G(x)]$:

(a) $G(0)$ is true;

(b) If $G(x)$ is true, then $G(Sx)$ is true.

Hence, by Gödel's completeness theorem:

(c) PA proves $[G(0)]$;

(d) PA proves $[G(x) \rightarrow G(Sx)]$.

> *Gödel's Completeness Theorem*: In any first-order predicate calculus, the theorems are precisely the logically valid well-formed formulas (*i. e. those that are true in every model of the calculus*).

Further, by Generalisation:

(e) PA proves $[(\forall x)(G(x) \rightarrow G(Sx))]$;

> *Generalisation in PA*: $[(\forall x)A]$ follows from $[A]$.

[41] For purposes of this investigation we may take this to be the first order predicate calculus $K$ as defined in [Me64], p.57.

Hence, by Induction:

(f) $[(\forall x)G(x)]$ is provable in PA.

*Induction Axiom Schema of PA*: For any formula $[F(x)]$ of PA:
$[F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(Sx)) \rightarrow (\forall x)F(x))]$

In other words, except 0, every element in the domain of any model of PA is a 'successor'. Further, the standard PA axioms ensure that $x$ can only be a 'successor' of a unique element in any model of PA.

Since Cantor's first limit ordinal $\omega$ is not the 'successor' of any ordinal in the sense required by the PA axioms, and since there are no infinitely descending sequences of ordinals[42] in a model—if any—of a first order set theory such as ZF, the theorem follows. □

### 4.B. Why we cannot force PA to admit a transfinite ordinal

Theorem 4.1 reflects the fact that we can define the usual order relation '$<$' in PA so that every instance of the PA Axiom Schema of Finite Induction, such as, say:

$(i)$ $[F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(Sx)) \rightarrow (\forall x)F(x))]$

yields the weaker PA theorem:

$(ii)$ $[F(0) \rightarrow ((\forall x)((\forall y)(y < x \rightarrow F(y)) \rightarrow F(x)) \rightarrow (\forall x)F(x))]$

Now, if we interpret PA without relativisation in ZF[43]— i.e., numerals as finite ordinals, $[Sx]$ as $[x \cup \{x\}]$, etc.— then $(ii)$ always translates in ZF as a theorem:

$(iii)$ $[F(0) \rightarrow ((\forall x)((\forall y)(y \in x \rightarrow F(y)) \rightarrow F(x)) \rightarrow (\forall x)F(x))]$

However, $(i)$ does not always translate similarly as a ZF-theorem, since the following is not necessarily provable in ZF:

$(iv)$ $[F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(x \cup \{x\})) \rightarrow (\forall x)F(x))]$

*Example*: Define $[F(x)]$ as '$[x \in \omega]$'.

We conclude that, whereas the language of ZF admits as a constant the first limit ordinal $\omega$ which would interpret in any putative model of ZF as the ('completed' infinite) set $\omega$ of all finite ordinals:

**Corollary 4.2.** *The language of PA admits of no constant that interprets in any model of PA as the set $N$ of all natural numbers.*

We note that it is the non-logical Axiom Schema of Finite Induction of PA which does not allow us to introduce—contrary to what is suggested by standard texts[44]—an 'actual' (*or 'completed'*) infinity disguised as an arbitrary constant (usually denoted by $c$ or $\infty$) into either the language, or a putative model, of PA[45].

[42] cf. [Me64], p.261.
[43] In the sense indicated by Feferman [Fe92].
[44] eg. [HP98], p.13, §0.29; [Ka91], p.11 & p.12, fig.1; [BBJ03]. p.306, Corollary 25.3; [Me64], p.112, Ex. 2.
[45] A possible reason why the Axiom Schema of Finite Induction does not admit non-finitary reasoning into either PA, or into any model of PA, is suggested in §6.A. below.

# 5. Forcing PA to admit denumerable descending dense sequences

The significance of Theorem 4.1 is seen in the next two arguments, which attempt to implicitly bypass the Theorem's constraint by appeal to the Compactness Theorem for forcing a non-standard model onto PA[46].

However, we argue in both cases that applying the Compactness Theorem constructively—even from a classical perspective—does not logically yield a non-standard model for PA with an 'infinite' integer as claimed[47].

## 5.A. An argument for a non-standard model of PA

The first is the argument[48] that we can define a non-standard model of PA with an infinite descending chain of successors, where the only non-successor is the null element 0:

1. Let $<N$ (*the set of natural numbers*); $=$ (*equality*); $S$ (*the successor function*); $+$ (*the addition function*); $*$ (*the product function*); 0 (*the null element*)$>$ be the structure that serves to define a model of PA, say $N$.

2. Let T[$N$] be the set of PA-formulas that are satisfied or true in $N$.

3. The PA-provable formulas form a subset of T[$N$].

4. Let $\Gamma$ be the countable set of all PA-formulas of the form $[c_n = Sc_{n+1}]$, where the index $n$ is a natural number.

5. Let T be the union of $\Gamma$ and T[$N$].

6. T[$N$] plus any finite set of members of $\Gamma$ has a model, e.g., $N$ itself, since $N$ is a model of any finite descending chain of successors.

7. Consequently, by Compactness, T has a model; call it $M$.

8. $M$ has an infinite descending sequence with respect to $S$ because it is a model of $\Gamma$.

9. Since PA is a subset of T, $M$ is a non-standard model of PA.

## 5.B. Why the argument in §5.A. is logically fragile

However if—as claimed in §5.A.(6) above—$N$ is a model of T[$N$] plus any finite set of members of $\Gamma$, and the PA term $[c_n]$ is well-defined for any given natural number $n$, then:

- All PA-formulas of the form $[c_n = Sc_{n+1}]$ are PA-provable,
- $\Gamma$ is a proper sub-set of the PA-provable formulas, and
- T is identically T[$N$].

[46] eg. [Ln08]; [Ka91], pp.10-11, p.74 & p.75, Theorem 6.4.
[47] And as suggested also by standard texts in such cases; eg. [BBJ03]. p.306, Corollary 25.3; [Me64], p.112, Ex. 2.
[48] [Ln08].

*Reason*: The argument cannot be that some PA-formula of the form $[c_n = Sc_{n+1}]$ is true in $N$, but not PA-provable, as this would imply that if PA is consistent then PA+$[\neg(c_n = Sc_{n+1})]$ has a model other than $N$; in other words, it would presume that which is sought to be proved, namely that PA has a non-standard model[49]!

Consequently, the postulated model $M$ of T in §5.A.(7) by 'Compactness' is the model $N$ that defines T$[N]$. However, $N$ has no infinite descending sequence with respect to $S$, even though it is a model of $\Gamma$.

Hence the argument does not establish the existence of a non-standard model of PA with an infinite descending sequence with respect to the successor function $S$.

## 5.C. A formal argument for a non-standard model of PA

The second is the more formal argument[50]:

> "Let $Th(\mathbb{N})$ denote the complete $\mathcal{L}_A$-theory of the standard model, i.e. $Th(\mathbb{N})$ is the collection of all true $\mathcal{L}_A$-sentences. For each $n \in \mathbb{N}$ we let $\underline{n}$ be the closed term $(\ldots(((1+1)+1)+\ldots+1)))(n\ 1s)$ of $\mathcal{L}_A$; $\underline{0}$ is just the constant symbol 0. We now expand our language $\mathcal{L}_A$ by adding to it a new constant symbol $c$, obtaining the new language $\mathcal{L}_c$, and consider the following $\mathcal{L}_c$-theory with axioms
>
> $\rho$ (for each $\rho \in Th(\mathbb{N})$)
>
> and
>
> $c > \underline{n}$ (for each $n \in \mathbb{N}$)
>
> This theory is consistent, for each finite fragment of it is contained in
>
> $T_k = Th(\mathbb{N}) \cup \{c > \underline{n} \mid n < k\}$
>
> for some $k \in \mathbb{N}$, and clearly the $\mathcal{L}_c$-structure $(\mathbb{N}, k)$ with domain $\mathbb{N}$, 0, 1, +, $\cdot$ and $<$ interpreted naturally, and $c$ interpreted by the integer $k$, satisfies $T_k$. Thus by the compactness theorem $\cup_{k\in\mathbb{N}} T_k$ is consistent and has a model $M_c$. The first thing to note about $M_c$ is that
>
> $M_c \models c > \underline{n}$
>
> for all $n \in \mathbb{N}$, and hence it contains an 'infinite' integer."

## 5.D. Why the argument in §5.C. too is logically fragile

We note again that, from an arithmetical perspective, any application of the Compactness Theorem to PA cannot ignore currently accepted computationalist doctrines of objectivity[51] and contradict the constructive assignment of satisfaction and truth to the atomic formulas of PA (therefore to the compound formulas under Tarski's inductive definitions) in terms of either algorithmical verifiability or algorithmic computability[52].

[49] To place this distinction in perspective, Adrien-Marie Legendre and Carl Friedrich Gauss independently conjectured in 1796 that, if $\pi(x)$ denotes the number of primes less than $x$, then $\pi(x)$ is asymptotically equivalent to $x/\text{In}(x)$. Between 1848/1850, Pafnuty Lvovich Chebyshev confirmed that if $\pi(x)/\{x/\text{In}(x)\}$ has a limit, then it must be 1. However, the crucial question of whether $\pi(x)/\{x/\text{In}(x)\}$ has a limit at all was answered in the affirmative using analytic methods independently by Jacques Hadamard and Charles Jean de la Vallée Poussin only in 1896, and using only elementary methods by Atle Selberg and Paul Erdös in 1949.

[50] [Ka91], pp.10-11; attributed as essentially Skolem's argument in [Sk34].

[51] cf. [Mu91].

[52] [An12], §3.

Accordingly, from an arithmetical perspective we can only conclude by the Compactness Theorem that if $Th(\mathbb{N})$ is the $\mathcal{L}_A$-theory of the standard model (interpretation), then we may consistently add to it the following as an additional—not necessarily independent—axiom:

$(\exists y)(y > x)$.

Moreover, even though $(\exists y)(y > x)$ is algorithmically computable as always true in the standard model—whence all instances of it are also therefore in $Th(\mathbb{N})$—we cannot conclude by the Compactness Theorem that $\cup_{k\in\mathbb{N}}T_k$ is consistent and has a model $M_c$ which contains an 'infinite' integer.

*Reason*: The condition '$k \in \mathbb{N}$' in $\cup_{k\in\mathbb{N}}T_k$ requires, first of all, that we must be able to extend $Th(\mathbb{N})$ by the addition of a 'relativised' axiom[53] such as:

$(\exists y)((x \in \mathbb{N}) \rightarrow (y > x))$,

from which we may conclude the existence of some $c$ such that:

$M_c \models c > \underline{n}$

for all $n \in \mathbb{N}$.

However, we shall then argue that even this would not yield a model for $Th(\mathbb{N})$, since every model of $Th(\mathbb{N})$ is by definition a model of (the provable formulas of) PA and, by Theorem 4.1, we cannot introduce a 'completed' infinity such as $\mathbb{N}$ into into either PA or any model of PA!

As the argument stands, it seeks to violate finitarity by adding a new constant $c$ to the language $\mathcal{L}_A$ of PA that is not definable in $\mathcal{L}_A$ and, ipso facto, adding an atomic formula $[c = x]$ to PA whose satisfaction under any interpretation of PA is not algorithmically verifiable!

Since the atomic formulas of PA are algorithmically verifiable under the standard interpretation[54], the above conclusion too postulates that which it seeks to prove!

Moreover, the postulation would be false if $Th(\mathbb{N})$ were categorical.

Since $Th(\mathbb{N})$ must have a non-standard model if it is not categorical, we consider next whether we may conclude from Gödel's incompleteness argument[55] that any such model can have an 'infinite' integer.

## 6. Gödel's argument for a non-standard model of PA

We begin by considering the Gödelian formula $[(\forall x)R(x)]$[56] which is unprovable in PA if PA is consistent, even though the formula $[R(n)]$ is provable in a consistent PA for any given PA numeral $[n]$.

Now, it follows from Gödel's reasoning[57] that:

**Theorem 6.1.** *If PA is consistent, then we may add the PA formula* $[\neg(\forall x)R(x)]$ *as an axiom to PA without inviting inconsistency.*

**Theorem 6.2.** *If PA is* $\omega$*-consistent, then we may add the PA formula* $[(\forall x)R(x)]$ *as an axiom to PA without inviting inconsistency.*

Gödel concluded from this that:

---

[53] cf. [Fe92]; [Me64], p.192.

[54] [An12], Corollary 2.

[55] In [Go31].

[56] In his seminal 1931 paper [Go31], Kurt Gödel defines, and refers to, the formula corresponding to $[R(x)]$ only by its 'Gödel' number $r$ (op. cit., p.25, Eqn.(12)), and to the formula corresponding to $[(\forall x)R(x)]$ only by its 'Gödel' number 17 *Gen* $r$.

[57] [Go31], p.25(1) & p.25(2).

**Corollary 6.3.** *If PA is $\omega$-consistent, then there are at least two distinctly different models of PA.* □

If we assume that a consistent PA is necessarily $\omega$-consistent, then it follows that one of the two putative models postulated by Corollary 6.3 must contain elements other than the natural numbers.

We conclude that Gödel's justification for the assumption that non-standard models of PA containing elements other than the natural numbers are logically feasible lies in his non-constructive—and logically fragile—assumption that a consistent PA is necessarily $\omega$-consistent.

## 6.A. Why Gödel's assumption is logically fragile

Now, whereas Gödel's proof of Corollary 6.3 appeals to the non-constructive Aristotle's particularisation, a constructive proof of the Corollary follows trivially from evidence-based interpretations of PA[58].

*Reason*: Tarski's inductive definitions allow us to provide *finitary* satisfaction and truth certificates to all atomic (and ipso facto to all compound) formulas of PA over the domain $N$ of the natural numbers in *two* essentially different ways:

(1) In terms of algorithmic verifiabilty[59]; and

(2) In terms of algorithmic computability[60].

That there can be even one, let alone two, logically sound and finitary assignments of satisfaction and truth certificates to both the atomic and compound formulas of PA was hitherto unsuspected!

Moreover, neither the putative 'algorithmically verifiable' model, nor the 'algorithmically computable' model, of PA defined by these finitary satisfaction and truth assignments contains elements other than the natural numbers.

**(a) Any algorithmically verifiable model of PA is necessarily over $\mathbb{N}$**

For instance if, in the first case, we assume that the algorithmically verifiable atomic formulas of PA determine an algorithmically verifiable model of PA over the domain $\mathbb{N}$ of the PA numerals, then such a putative model would be isomorphic to the standard model of PA over the domain $N$ of the natural numbers[61].

However, such a putative model of PA over $\mathbb{N}$ would not be finitary since, if the formula $[(\forall x)F(x)]$ were to interpret as true in it, then we could only conclude that, for any numeral $[n]$, there is an algorithm which will finitarily certify the formula $[F(n)]$ as true under an algorithmically verifiable interpretation in $\mathbb{N}$.

We could not conclude that there is a single algorithm which, for any numeral $[n]$, will finitarily certify the formula $[F(n)]$ as true under the algorithmically verifiable interpretation in $\mathbb{N}$.

Consequently, even though the PA Axiom Schema of Finite Induction can be shown to interpret as true under the algorithmically verifiable interpretation of PA over the domain $\mathbb{N}$ of the PA numerals, the interpretation would not define a finitary model of PA.

However, if we were to assume that the algorithmically verifiable interpretation of PA defines a non-finitary model of PA, then it would follow that:

- PA is necessarily $\omega$-consistent;

[58] [An12].
[59] [An12], §4.2.
[60] [An12], §4.3.
[61] [An12], §4.2 & §5, Corollary 2.

- Aristotle's particularisation holds over $N$; and
- The 'standard' interpretation of PA also defines a non-finitary model of PA over $N$.

**(b) The algorithmically computable interpretation of PA is over $\mathbb{N}$**

The second case is where the algorithmically computable atomic formulas of PA determine an algorithmically computable model of PA over the domain $N$ of the natural numbers[62].

The algorithmically computable model of PA is finitary since we can show that, if the formula $[(\forall x)F(x)]$ interprets as true under it, then we may always conclude that there is a single algorithm which, for any numeral $[n]$, will finitarily certify the formula $[F(n)]$ as true in $N$ under the algorithmically computable interpretation.

Consequently we can show that all the PA axioms—including the Axiom Schema of Finite Induction—interpret finitarily as true in $N$ under the algorithmically computable interpretation of PA, and the PA Rules of Inference preserve such truth finitarily[63].

Thus the algorithmically computable interpretation of PA defines a finitary model of PA from which we may conclude that:

- PA is consistent[64].

### 6.B. Why we cannot conclude that PA is necessarily $\omega$-consistent

By the way the above finitary interpretation (b) is defined under Tarski's inductive definitions[65], if a PA-formula $[F]$ interprets as true in the corresponding finitary model of PA, then there is an algorithm that provides a certificate for such truth for $[F]$ in $N$; whilst if $[F]$ interprets as false in the above finitary model of PA, then there is no algorithm that can provide such a truth certificate for $[F]$ in $N$[66].

Now, if there is no algorithm that can provide such a truth certificate for the Gödelian formula $[R(x)]$ in $N$, then we would have by definition first that the PA formula $[\neg(\forall x)R(x)]$ is true in the model, and second by Gödel's reasoning that the formula $[R(n)]$ is true in the model for any given numeral $[n]$. Hence Aristotle's particularisation would not hold in the model.

However, by definition if PA were $\omega$-consistent then Aristotle's particularisation must necessarily hold in every model of PA.

It follows that unless we can establish that there is some algorithm which can provide such a truth certificate for the Gödelian formula $[R(x)]$ in $N$, we cannot make the unqualified assumption—as Gödel appears to do—that a consistent PA is necessarily $\omega$-consistent.

## Conclusion

We have argued that standard arguments for the existence of non-standard models of the first-order Peano Arithmetic PA with domains other than the domain $N$ of the natural numbers should be treated as logically fragile even from within classical logic. In particular we have argued that although Gödel's argument for the existence of a non-standard model of PA does yield a model of PA other than the classical non-finitary 'standard' model, we cannot conclude from it that the domain is other than the domain $N$ of the natural numbers.

[62] [An12], §4.3 & §5.2.
[63] [An12], §5.2 Theorem 4.
[64] [An12], §5.3, Theorem 6.
[65] [An12], §4.3.
[66] [An12], §2.

# References


[BBJ03] George S. Boolos, John P. Burgess, Richard C. Jeffrey. 2003. *Computability and Logic* (4th ed). Cambridge University Press, Cambridge.

[Be59] Evert W. Beth. 1959. *The Foundations of Mathematics.* Studies in Logic and the Foundations of Mathematics. Edited by L. E. J. Brouwer, E. W. Beth, A. Heyting. 1959. North Holland Publishing Company, Amsterdam.

[BF58] Paul Bernays and Abraham A. Fraenkel. 1958. *Axiomatic Set Theory.* Studies in Logic and the Foundations of Mathematics. Edited by L. E. J. Brouwer, E. W. Beth, A. Heyting. 1959. North Holland Publishing Company, Amsterdam.

[Bo00] Andrey I. Bovykin. 2000. *On order-types of models of arithmetic.* Thesis submitted to the University of Birmingham for the degree of Ph.D. in the Faculty of Science, School of Mathematics and Statistics, The University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom, 13 April 2000.

[Br08] L. E. J. Brouwer. 1908. *The Unreliability of the Logical Principles.* English translation in A. Heyting, Ed. L. E. J. Brouwer: Collected Works 1: *Philosophy and Foundations of Mathematics.* Amsterdam: North Holland / New York: American Elsevier (1975): pp. 107-111.

[Co66] Paul J. Cohen. 1966. *Set Theory and the Continuum Hypothesis.* (Lecture notes given at Harvard University, Spring 1965) W. A. Benjamin, Inc., New York.

[Da82] Martin Davis. 1958. *Computability and Unsolvability.* 1982 ed. Dover Publications, Inc., New York.

[EC89] Richard L. Epstein, Walter A. Carnielli. 1989. *Computability: Computable Functions, Logic, and the Foundations of Mathematics.* Wadsworth & Brooks, California.

[Fe92] Solomon Feferman. 1992. *What rests on what: The proof-theoretic analysis of mathematics*. Invited lecture, 15th International Wittgenstein Symposium: *Philosophy of Mathematics*, held in Kirchberg/Wechsel, Austria, 16-23 August 1992.

[Go31] Kurt Gödel. 1931. *On formally undecidable propositions of Principia Mathematica and related systems I.* Translated by Elliott Mendelson. In M. Davis (ed.). 1965. *The Undecidable.* Raven Press, New York.

[HA28] David Hilbert & Wilhelm Ackermann. 1928. *Principles of Mathematical Logic.* Translation of the second edition of the *Grundzüge Der Theoretischen Logik.* 1928. Springer, Berlin. 1950. Chelsea Publishing Company, New York.

[Hi25] David Hilbert. 1925. *On the Infinite.* Text of an address delivered in Münster on 4th June 1925 at a meeting of the Westphalian Mathematical Society. In Jean van Heijenoort. 1967. Ed. *From Frege to Gödel: A source book in Mathematical Logic, 1878 - 1931.* Harvard University Press, Cambridge, Massachusetts.

[HP98] Petr Hájek and Pavel Pudlák. 1998. *Metamathematics of First-Order Arithmetic.* Volume 3 of *Perspectives in Mathematical Logic* Series, ISSN 0172-6641. Springer-Verlag, Berlin, 1998.

[Ha60] G. H. Hardy. 1960. *A Course of Pure Mathematics.* Tenth edition. Cambridge University Press, Cambridge, U. K.

[Ka91] Richard Kaye. 1991. *Models of Peano Arithmetic.* Oxford Logic Guides, 15. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[Ka11] Richard Kaye. 2011. *Tennenbaum's theorem for models of arithmetic.* In *Set Theory, Arithmetic, and Foundations of Mathematics.* Eds. Juliette Kennedy & Roman Kossak. Lecture Notes in Logic (No. 36). pp.66-79. Cambridge University Press, 2011. Cambridge Books Online. http://dx.doi.org/10.1017/CBO9780511910616.005.

[Kl52] Stephen Cole Kleene. 1952. *Introduction to Metamathematics.* North Holland Publishing Company, Amsterdam.

[Kn63] G. T. Kneebone. 1963. *Mathematical Logic and the Foundations of Mathematics: An Introductory Survey.* D. Van Norstrand Company Limited, London.

[Ko06] Roman Kossak and James H. Schmerl. 2006. *The structure of models of Peano arithmetic.* Oxford Logic Guides, 50. Oxford Science Publications. The Clarendon Press, Oxford University Press, Oxford, 2006.

[La29] Edmund Landau. 1929. *Foundations of Analysis: The Arithmetic of Whole, Rational, Irrational and Complex Numbers.* Translated by F. Steinhardt. 2nd. ed. 1960. Chelsea Publishing Company, New York, N. Y.

[Li64] A. H. Lightstone. 1964. *The Axiomatic Method.* Prentice Hall, NJ.

[Ln08] Laureano Luna. 2008. *On non-standard models of Peano Arithmetic*. The Reasoner, Vol(2)2 p7.

[Me64] Elliott Mendelson. 1964. *Introduction to Mathematical Logic.* Van Norstrand, Princeton.

[Mu91] Chetan R. Murthy. 1991. *An Evaluation Semantics for Classical Proofs.* Proceedings of Sixth IEEE Symposium on Logic in Computer Science, pp. 96-109, (also Cornell TR 91-1213), 1991.

[Nv64] P. S. Novikov. 1964. *Elements of Mathematical Logic.* Oliver & Boyd, Edinburgh and London.

[Qu63] Willard Van Orman Quine. 1963. *Set Theory and its Logic.* Harvard University Press, Cambridge, Massachusette.

[Rg87] Hartley Rogers Jr. 1987. *Theory of Recursive Functions and Effective Computability.* MIT Press, Cambridge, Massachusetts.

[Ro53] J. Barkley Rosser. 1953. *Logic for Mathematicians.* McGraw Hill, New York.

[Sh67] Joseph R. Shoenfield. 1967. *Mathematical Logic*. Reprinted 2001. A. K. Peters Ltd., Massachusetts.

[Sk28] Thoralf Skolem. 1928. *On Mathematical Logic*. Text of a lecture delivered on 22nd October 1928 before the Norwegian Mathematical Association. In Jean van Heijenoort. 1967. Ed. *From Frege to Gödel: A source book in Mathematical Logic, 1878 - 1931*. Harvard University Press, Cambridge, Massachusetts.

[Sk34] Thoraf A. Skolem. 1934. *Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen*. Fundamenta Mathematicae, 23, 150-161. English version: *Peano's axioms and models of arithmetic*. In *Mathematical interpretations of formal systems*. North Holland, Amsterdam, 1955, pp.1-14.

[Sm92] Raymond M. Smullyan. 1992. *Gödel's Incompleteness Theorems*. Oxford University Press, Inc., New York.

[Su60] Patrick Suppes. 1960. *Axiomatic Set Theory*. Van Norstrand, Princeton.

[Tu36] Alan Turing. 1936. *On computable numbers, with an application to the Entscheidungsproblem*. In M. Davis (ed.). 1965. *The Undecidable*. Raven Press, New York. Reprinted from the Proceedings of the London Mathematical Society, ser. 2. vol. 42 (1936-7), pp.230-265; corrections, Ibid, vol 43 (1937) pp. 544-546.

[Wa63] Hao Wang. 1963. *A survey of Mathematical Logic*. North Holland Publishing Company, Amsterdam.

[An12] Bhupinder Singh Anand. 2012. *Evidence-Based Interpretations of PA*. Proceedings of the Symposium on Computational Philosophy at the AISB/IACAP World Congress 2012-Alan Turing 2012, 2-6 July 2012, University of Birmingham, Birmingham, UK.

[An13] . . . . 2013. *A suggested mathematical perspective for the EPR argument.* Presented on 7'th April at the workshop on '*Logical Quantum Structures*' at UNILOG'2013, $4^{th}$ World Congress and School on Universal Logic, $29^{th}$ March 2013 - $7^{th}$ April 2013, Rio de Janeiro, Brazil.

[An15] . . . . 2015. *Why Hilbert's and Brouwer's interpretations of quantification are complementary and not contradictory.* Presented on $10^{th}$ June at the Epsilon 2015 workshop on '*Hilbert's Epsilon and Tau in Logic, Informatics and Linguistics*', $10^{th}$ June 2015 - $12^{th}$ June 2015, University of Montpellier, France.

[An15a] . . . . 2015. *Algorithmically Verifiable Logic vis à vis Algorithmically Computable Logic: Could resolving EPR need two complementary Logics?* Presented on 26'th June at the workshop on '*Emergent Computational Logics*' at UNILOG'2015, $5^{th}$ World Congress and School on Universal Logic, $20^{th}$ June 2015 - $30^{th}$ June 2015, Istanbul, Turkey.

Bhupinder Singh Anand
#1003 B Wing, Lady Ratan Tower
Dainik Shivner Marg
Gandhinagar, Worli
Mumbai - 400 018
Maharashtra, India.
*e-mail*: bhup.anand@gmail.com
*Personal archive*: http://alixcomsi.com/Index01.pdf.
*On-going work pages*: http://foundationalperspectives.wordpress.com/