Algorithmically Verifiable Logic *vis à vis* Algorithmically Computable Logic

Could resolving *EPR* need two complementary Logics?

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Abstract. We suggest that the paradoxical element which surfaced as a result of the *EPR* argument (due to the perceived conflict implied by Bell's inequality between the, seemingly essential, non-locality required by current interpretations of Quantum Mechanics, and the essential locality required by current interpretations of Classical Mechanics) may reflect merely lack of recognition that any mathematical language which can adequately express and effectively communicate the laws of nature may be consistent under two, essentially different but complementary and not contradictory, logics for assigning truth values to the propositions of the language, such that the latter are capable of representing—as deterministic—the unpredictable characteristics of quantum behaviour. We show how the anomaly may dissolve if a physicist could cogently argue that: (i) All properties of physical reality are deterministic, but not necessarily mathematically predetermined—in the sense that any physical property could have one, and only one, value at any time $t(n)$, where the value is completely determined by some natural law which need not, however, be representable by algorithmically computable expressions (and therefore be mathematically predictable). (ii) There are elements of such a physical reality whose properties at any time $t(n)$ are determined completely in terms of their putative properties at some earlier time $t(0)$. Such properties are predictable mathematically since they are representable by algorithmically computable functions. The values of any two such functions with respect to their variables are, by definition, independent of each other and must, therefore, obey Bell's inequality. The Laws of Classical Mechanics determine the nature and behaviour of such physical reality only, and circumscribe the limits of reasoning and cognition in any emergent mechanical intelligence. (iii) There could be elements of such a physical reality whose properties at any time $t(n)$ cannot be theoretically determined completely from their putative properties at some earlier time $t(0)$. Such properties are unpredictable mathematically since they are only representable mathematically by algorithmically verifiable, but not algorithmically computable, functions. The values of any two such functions with respect to their variables may, by definition, be dependent on each other and need not, therefore, obey Bell's inequality. The Laws of Quantum Mechanics determine the nature and behaviour of such physical reality, and circumscribe the limits of reasoning and cognition in any emergent humanlike intelligence. In this paper we formally define the common language, but distinctly different logics, of such functions and suggest a perspective from which to view the anomalous philosophical issues underlying some current concepts of quantum phenomena such as indeterminacy, fundamental dimensionless constants, conjugate properties, uncertainty, entanglement, EPR paradox, Bell's inequalities, and Schrödinger's cat paradox. We also briefly indicate the significance of such a perspective for defining the concept of emergence in a mechanical intelligence.

Contents

1. Introduction

We shall simplify the perspective of this exploratory paper by making an arbitrary distinction between the three disciplines:

- ***Applied science***, whose concern is our sensory observations of a 'common' external world;
- ***Philosophy***, whose concern is abstracting a coherent perspective of the external world from our sensory observations; and
- ***Mathematics***, whose concern is adequately expressing such abstractions in a formal language of unambiguous communication.

In what follows, our concern is only that of mathematics, where we define:

Definition 1. *A finite set λ of rules is a Logic of a formal mathematical language $\mathcal{L}$ if, and only if, λ constructively assigns unique truth-values:*

> ***(a)*** *Of provability/unprovability to the formulas of $\mathcal{L}$; and*
>
> ***(b)*** *Of truth/falsity to the sentences of the Theory $T(\mathcal{U})$ which is defined semantically by the λ-interpretation of $\mathcal{L}$ over a structure $\mathcal{U}$.*

We shall now show that such a definitional approach to 'logic' and 'truth' allows us to:

- Equate the provable formulas of the first order Peano Arithmetic PA with the PA formulas that are 'true' under an algorithmically computable interpretation of PA over the structure $\mathbb{N}$ of the natural numbers (Theorem 6.1);
- Adequately represent some of the philosophically troubling abstractions of the physical sciences mathematically;

- Interpret such representations unambiguously; and
- Conclude further:

  • First (Thesis 7) that the concept of infinity is an emergent feature of any mechanical intelligence whose true arithmetical propositions are provable in the first-order Peano Arithmetic in the sense of Theorem 6.1; and

  • Second (Thesis 8) that discovery and formulation of the laws of quantum physics lies within the algorithmically computable logic and reasoning of a mechanical intelligence whose logic is circumscribed by the first-order Peano Arithmetic.

1.A. The EPR paradox

For instance, amongst the philosophically disturbing features of the standard Copenhagen interpretation of Quantum Theory are its essential indeterminateness[1] (highlighed famously by Erwin Schrödinger's caustic observation regarding the philosophical consequences of the proposed mathematical interpretation of the ψ-function if taken to imply that the objective state of nature is essentially probabilistic[2]) and its essential separation of the world into 'system' and 'observer'[3].

In 1935 Albert Einstein, Boris Podolsky and Nathan Rosen noted[4] that accepting Quantum Theory, but denying these features of the Copenhagen interpretation, logically entails accepting:

- Either that the world is non-local[5] (thus contradicting Special Relativity);
- Or that there are hidden variables[6] which would eliminate the need for accepting these features as necesary to any sound interpretation of Quantum Theory.

In 1952 David Bohm proposed[7] an alternative mathematical development of the existing Quantum Theory, which was essentially equivalent to it but based on Louis de Broglie's pilot wave theory.

However, even though Bohm's interpretation eliminated the need for indeterminism and the separation of the world into 'system' and 'observer', it appealed unappealingly to hidden variables[8] and, presumably, hidden natural laws that—we may reasonably presume further—were implicitly assumed by Bohm to be representable in principle by well-defined classically computable mathematical functions[9].

Moreover, in 1964 John Stewart Bell showed[10] that any interpretation of Quantum Theory which appeals to hidden variables and (again, presumably classically computable) functions in the above sense must necessarily be non-local.

[1] "It is a general principle of orthodox formulations of quantum theory that measurements of physical quantities do not simply reveal pre-existing or predetermined values, the way they do in classical theories. Instead, the particular outcome of the measurement somehow "emerges" from the dynamical interaction of the system being measured with the measuring device, so that even someone who was omniscient about the states of the system and device prior to the interaction couldn't have predicted in advance which outcome would be realized". [Sh+11].

[2] "One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The ψ-function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts". [Sc35], §5.

[3] cf. [Sh+11].

[4] [EPR35].

[5] "'Non-local' ... means that there exist interactions between events that are too far apart in space and too close together in time for the events to be connected even by signals moving at the speed of light". [Sh+11].

[6] "Traditionally, the phrase 'hidden variables' is used to characterize any elements supplementing the wave function of orthodox quantum theory". [Sh+11].

[7] [Bo52].

[8] "This terminology is, however, particularly unfortunate in the case of the de Broglie-Bohm theory, where it is in the supplementary variables—definite particle positions—that one finds an image of the manifest world of ordinary experience". [Sh+11].

[9] Which could be considered as having pre-existing or predetermined mathematical values over the domain over which the functions are well-defined.

[10] [Bl64].

However, our foundational investigations into the (apparently unrelated) area of evidence-based and finitary interpretations of the first order Peano Arithmetic PA [11] now suggest that:

- If our above presumption concerning an implicit appeal by Bohm and Bell to functions that are implicitly assumed to be classically computable is correct,
- Then the hidden variables in the Bohm-de Broglie interpretation of Quantum Theory could as well be presumed to involve natural laws which are mathematically representable only by functions that are algorithmically verifiable, but not algorithmically computable[12],
- In which case Bohm's interpretation might avoid being held as admitting 'non-locality' by Bell's reasoning.

1.B. The underlying perspective of this thesis

The underlying perspective of this thesis is that:

> (1) Classical physics assumes that all the observable laws of nature can be mathematically represented in terms of well-defined functions that are algorithmically computable (as defined in the next section).
>
> > Since the functions are well-defined, their values are pre-existing and predetermined as mappings that are capable of being known in their infinite totalities to an omniscient intelligence, such as Laplace's *intellect*[13].
>
> (2) However, the overwhelming experimental verification of the mathematical predictions of Quantum Theory suggests that the actual behavior of the real world cannot be assumed as pre-existing and predetermined in this sense.
>
> > In other words, the consequences of some experimental interactions are theoretically incapable of being completely known in advance even to an omniscient intelligence, such as Laplace's *intellect*.
> >
> > So all the observable laws of nature cannot be represented mathematically in terms of functions that are algorithmically computable (as defined in the next section).
>
> (3) It follows that:
>
> - Either there is no way of representing all the observable laws of nature mathematically in a deterministic model;
> - Or all the observable laws of nature *can* be represented mathematically in a deterministic model—but in terms of functions that are 'computable' in a non-algorithmic sense (we define these in the next section as algorithmically verifiable functions).
>
> (4) The Copenhagen interpretation appears to opt for the first option in (3) above, and hold that there is no way of representing all the observable laws of nature mathematically in a deterministic model.
>
> > In other words, the interpretation is not overly concerned with the seemingly essential non-locality of Quantum Theory, and its conflict with the deterministic mathematical representation of the laws of Special Relativity.
>
> (5) The Bohm-de Broglie interpretation appears to reject the first option in (3) above, and to propose a way of representing all the observable laws of nature mathematically in a deterministic model and, presumably, in terms of functions that are taken implicitly to be algorithmically computable.

[11] [An12].

[12] As defined below—hence mathematically determinate but unpredictable.

[13] "We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all the forces that animate Nature and the mutual positions of the beings that comprise it, if this intellect were vast enough to submit its data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom: for such an intellect nothing could be uncertain; and the future just like the past would be present before its eyes."—Pierre Simon Laplace, A Philosophical Essay on Probabilities.

However, the Bohm-de Broglie interpretation has not so far been viewed as being capable of mathematically avoiding the seemingly essential non-local feature of Quantum Theory implied by Bell's inequalities.

(6) In this paper we propose the second option in (3) above; i.e., that the apparently non-local feature of Quantum Theory may actually be indicative of a non-constructive and 'counter intuitive-to-human-intelligence' phenomena in nature that could, however, be mathematically represented by functions that:

- Are algorithmically verifiable (Definition 3);
- But not algorithmically computable (Definition 4).

2. The EPR paradox reflects an implicit mathematical ambiguity in interpreting quantification

We shall now argue that the EPR paradox is essentially a mathematical argument whose paradoxical conclusion merely reflects an implicit mathematical ambiguity in interpreting quantification[14], and whose roots lie in the assumption of conventional Gödelian wisdom that:

- The 'true' sentences of a theory $T(\mathcal{U})$ cannot be defined algorithmically by any logic of the formal language $\mathcal{L}$ of the theory $T(\mathcal{U})$,
- But are an essential feature of the structure $\mathcal{U} =< A, \alpha >$,
- Which is defined by a non-empty domain A, and an algebra α defined over A.

However, we hold that such a non-constructive perspective implicitly implies that the concept of 'truth' must then 'exist' Platonically, in the sense of needing to be discovered by some witness-dependent means—eerily akin to a 'revelation'—if the domain A is infinite.

2.A. Truth-values must be a computational convention

In this investigation we therefore adopt the constructive perspective that:

- The 'true' sentences of a theory $T(\mathcal{U})$ must be defined as objective assignments,
- By a computational convention that is witness-independent,
- In terms of the Tarskian 'satisfaction' and 'truth' of the corresponding formulas, over the structure $\mathcal{U}$,
- Of the formal language $\mathcal{L}$ of $T(\mathcal{U})$ under a constructive interpretation.

2.B. Distinguishing between *For any* and *For all*

However, we are then faced with the ambiguity where, if $F^*(x)$ is the interpretation in $\mathcal{U}$ of the $\mathcal{L}$-formula $[F(x)]$:

- Is the formula $[(\forall x)F(x)]$ of a formal language $\mathcal{L}$ to be interpreted constructively as:
  - '*For any* a, $F^*(a)$',
  - Which holds if, and only if,
  - For *any specified* element a of the domain A,
  - There is *algorithmic evidence* that $F^*(a)$ holds in $\mathcal{U}$?
- Or is $[(\forall x)F(x)]$ to be interpreted finitarily as:

[14] See also [An15].

  - ‘*For all* a, $F^*(a)$’,
  - Which holds if, and only if,
  - There is *algorithmic evidence* that,
  - For *any specified* element a of A, $F^*(a)$ holds in $\mathcal{U}$?

Where:

Definition 2. *An element a of a domain A is defined as specifiable in the language $\mathcal{L}$ of a structure $\mathcal{U}$ if, and only if, it can be explicitly denoted as an $\mathcal{L}$-term by an $\mathcal{L}$-formula that interprets as an algorithmically computable constant in $\mathcal{U}$.*

2.C. Evidence-based reasoning

Keeping the need for such a distinction in mind, we note that Tarski's classic definitions now:

- Permit an intelligence,
  - Whether human,
  - Or mechanistic,
- To admit *TWO*
  - Finitary,
  - Evidence-based,
  - Inductive,
- Logics for assigning
  - The values of satisfaction and truth,
  - To the *atomic* formulas of $\mathcal{L}$,
  - Over the domain A,
- In *TWO,* essentially different, ways:
  (a) In terms of constructive algorithmic verifiabilty; and
  (b) In terms of finitary algorithmic computability.

3. A finitary perspective over the structure $\mathbb{N}$

The perspective we choose for addressing these issues is that of the structure $\mathbb{N}$, defined by:

- $\{N$ (*the set of natural numbers*);
- $=$ (*equality*);
- S (*the successor function*);
- $+$ (*the addition function*);
- $*$ (*the product function*);
- 0 (*the null element*)$\}$

which serves for a definition of today's standard interpretation, say $\boldsymbol{M}$, of the first-order Peano Arithmetic PA.

Our reason for choosing PA as the basis for our perspective is that—as we show in the second part of this paper—PA is a mathematical language of both adequate expression (Corollary 6.2) and effective communication (Theorem 6.1) which can provide the sound foundation (Theorem 6.3) needed by any computational language in which mechanical artefacts record their observations—of a putative 'common' external world—that reflect and extend what is directly experienced, or conjectured as indirectly observable, by our sensory perceptions, and in which applied science attempts to mathematically model the putative laws of nature that such observations suggest.

3.A. Algorithmic verifiability

We can express the distinction of Section 2.B. formally so that if, for instance, a PA formula $[(\forall x)F(x)]$ is to be interpreted constructively, as:

'For any x, $F^*(x)$'

over the structure $\mathbb{N}$ of the natural numbers, then the latter must be read consistently as[15]:

'$F^*(x)$ is algorithmically verifiable'

where:

Definition 3. *A number-theoretical relation $F^*(x)$ is algorithmically verifiable if, and only if, for any given natural number n, there is an algorithm $AL_{(F,\ n)}$ which can provide objective evidence[16] for deciding the truth/falsity of each proposition in the finite sequence $\{F^*(1), F^*(2), \ldots, F^*(n)\}$.*

3.B. Algorithmic computability

Whereas if $[(\forall x)F(x)]$ is intended to be interpreted constructively as:

'For all x, $F^*(x)$',

then the latter must be read consistently as:

'$F^*(x)$ is algorithmically computable'

where:

Definition 4. *A number theoretical relation $F^*(x)$ is algorithmically computable if, and only if, there is an algorithm AL_F that can provide objective evidence for deciding the truth/falsity of each proposition in the denumerable sequence $\{F^*(1), F^*(2), \ldots\}$.*

We note that:

- Algorithmic computability implies the existence of an algorithm that can decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions[17];
- Whereas algorithmic verifiability does not imply the existence of an algorithm that can decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions.

3.C. Functions that are algorithmically verifiable but not algorithmically computable

From the point of view of a finitary mathematical philosophy—which is the constraint within which an applied science ought to ideally operate—the significant difference between the two concepts could be expressed by saying that:

[15] cf. [An12].

[16] cf. [Mu91]: "It is by now folklore ... that one can view the *values* of a simple functional language as specifying *evidence* for propositions in a constructive logic ...".

[17] We note that the concept of 'algorithmic computability' is essentially an expression of the more rigorously defined concept of 'realizability' in [Kl52], p.503.

> We may treat the decimal representation of a real number as corresponding to a physically measurable limit[18]—and not only to a mathematically definable limit—if and only if such representation is definable by an algorithmically computable function.

We note that although every algorithmically computable relation is algorithmically verifiable, the converse is not true.

Theorem 3.1. *There are mathematical functions that are algorithmically verifiable but not algorithmically computable.*

Proof: (a) Since any real number is mathematically definable as the limit of a Cauchy sequence of rational numbers:

- Let $R(n)$ denote the n^{th} digit in the decimal expression of the real number R in binary notation.

- Then, for any given natural number n, there is an algorithm $AL_{(R,\ n)}$ that can decide the truth/falsity of each proposition in the finite sequence:

  $\{R(1) = 0, R(2) = 0, \ldots, R(n) = 0\}$.

- Hence, for any real number R, the relation $R(x) = 0$ is algorithmically verifiable trivially.

(b) Since it follows from Alan Turing's Halting argument[19] that there are algorithmically uncomputable real numbers:

- Let $[R(n)]$ denote the n^{th} digit in the decimal expression of an algorithmically *uncomputable* real number R in binary notation.

- By (a), the relation $[R(x) = 0]$ is algorithmically verifiable trivially.

- However, by definition there is no algorithm AL_R that can decide the truth/falsity of each proposition in the denumerable sequence:

  $\{[R(1) = 0], [R(2) = 0], \ldots\}$.

- Hence the relation $[R(x) = 0]$ is algorithmically verifiable but not algorithmically computable. □

3.D. Chaitin's constants are definable only by algorithmically verifiable but not algorithmically computable functions

We note that:

> (i) All the mathematically defined functions known to, and used by, science are algorithmically computable, including those that define transcendental numbers such as π, e, etc. They can be computed algorithmically as they are all definable as the limit of some well-defined infinite series of rationals.
>
> (ii) The existence of mathematical constants that are defined by functions which are algorithmically verifiable but not algorithmically computable—suggested most famously by Georg Cantor's diagonal argument—has been a philosophically debatable deduction.
>
> Such existential deductions have been viewed with both suspicion and scepticism by scientists such as Henri Poincaré, L. E. J. Brouwer, etc., and disputed most vociferously on philosophical grounds by Ludwig Wittgenstein[20].
>
> (iii) A constructive definition of an arithmetical Boolean function $[R(x)]$ that is true—hence algorithmically verifiable—but not provable in Peano Arithmetic—hence algorithmically not computable

[18] In the sense of a physically 'completable' infinite sequence (as needed to resolve Zeno's paradox).
[19] [Tu36], p.132, §8.
[20] [Wi78].

(Corollary 6.6)—was given by Kurt Gödel in his 1931 paper on formally undecidable arithmetical propositions[21]

(iv) The definition of a number-theoretic function that is algorithmically verifiable but not algorithmically computable was also given by Alan Turing in his 1936 paper on computable numbers[22].

He defined a halting function, say $H(n)$, that is 0 if, and only if, the Turing machine with code number n halts on input n. Such a function is mathematically well-defined, but assuming that it defines an algorithmically computable real number leads to a contradiction, Turing concluded the mathematical existence of algorithmically uncomputable real numbers.

(v) A definition of a number-theoretic function that is algorithmically verifiable but not algorithmically computable was given by Gregory Chaitin[23]; he defined a class of constants—denoted by Ω—which is such that if $C(n)$ is the n^{th} digit in the decimal expression of an Ω constant, then the function $C(x)$ is algorithmically verifiable but not algorithmically computable.

4. Physical constants representable by real numbers that are algorithmically verifiable but not algorithmically computable

Similarly, some physical constants may be representable by real numbers which are definable only by algorithmically verifiable but not algorithmically computable functions.

This is suggested by the following perspective of one of the challenging issues in physics, which seeks to theoretically determine the magnitude of some fundamental dimensionless constants:

> "... the numerical values of dimensionless physical constants are independent of the units used. These constants cannot be eliminated by any choice of a system of units. Such constants include:
>
> - α, the fine structure constant, the coupling constant for the electromagnetic interaction ($\approx 1/137.036$). Also the square of the electron charge, expressed in Planck units. This defines the scale of charge of elementary particles with charge.
> - μ or β, the proton-to-electron mass ratio, the rest mass of the proton divided by that of the electron (≈ 1836.15). More generally, the rest masses of all elementary particles relative to that of the electron.
> - α_s, the coupling constant for the strong force (≈ 1)
> - αG, the gravitational coupling constant ($\approx 10^{-38}$) which is the square of the electron mass, expressed in Planck units. This defines the scale of the mass of elementary particles.
>
> At the present time, the values of the dimensionless physical constants cannot be calculated; they are determined only by physical measurement. This is one of the unsolved problems of physics. ...
>
> The list of fundamental dimensionless constants decreases when advances in physics show how some previously known constant can be computed in terms of others. A long-sought goal of theoretical physics is to find first principles from which all of the fundamental dimensionless constants can be calculated and compared to the measured values. A successful 'Theory of Everything' would allow such a calculation, but so far, this goal has remained elusive."
>
> ... Dimensionless physical constant - Wikipedia

From the perspective of Section 3., we could now suggest that:

Thesis 1. *Some of the dimensionless physical constants are only representable in a mathematical language as real numbers that are defined by functions which are algorithmically verifiable, but not algorithmically computable.*

[21] [Go31]. It was also disputed vociferously by Wittgenstein ([Wi78]).
[22] [Tu36].
[23] [Ct82].

In other words, we cannot treat such constants as denoting—even in principle—a measurable limit, as we could a constant that is representable mathematically by a real number that is definable by algorithmically computable functions.

Completed Infinities

From the point of view of mathematical philosophy, this distinction would be intuitively expressed by the assertion that:

- Whilst a symbol for an 'unmeasurable' physical constant may be introduced into a physical theory as a primitive term without inviting inconsistency in the theory, the sequence of digits in the decimal representation of the 'measure' of an 'unmeasurable' physical constant *cannot* be treated in the mathematical language of the theory as a 'completed' infinite sequence;
- Whereas the corresponding sequence in the decimal representation of the 'measure' of a 'measurable' physical constant, when introduced as a primitive term into a physical theory, *can* be treated as a 'completed' infinite sequence in the mathematical language of the theory without inviting inconsistency.

4.A. Zeno's argument: The dichotomy between a continuous physical reality and its discretely representable mathematical theory

We note that Zeno's paradoxical arguments highlight the philosophical and theological dichotomy between our essentially 'continuous' perception of the physical reality that we seek to capture with our measurements, and the essential 'discreteness' of any mathematical language of Arithmetic in which we seek to express such measurements.

The distinction between algorithmic verifiability and algorithmic computability of Arithmetical functions could be seen as reflecting the dichotomy mathematically.

4.B. Classical laws of nature

For instance, the distinction suggests that classical mechanics *could* be held as complete with respect to the algorithmically computable representation of the physical world, in the sense that:

Thesis 2. *Classical laws of nature determine the nature and behaviour of all those properties of the physical world which are mathematically describable completely at any moment of time* $t(n)$ *by algorithmically computable functions from a given initial state at time* $t(0)$.

4.C. Neo-classical laws of nature

On the other hand, the distinction also suggests that:

Thesis 3. *Neo-classical laws of nature determine the nature and behaviour of those properties of the physical world which are describable completely at any moment of time* $t(n)$ *by algorithmically verifiable functions; however such properties are not completely describable by algorithmically computable functions from any given initial state at time* $t(0)$.

Since such behaviour[24] follows fixed laws and is determinate (even if not algorithmically predictable by classical laws), Albert Einstein could have been justified in his belief that:

> "...God doesn't play dice with the world"

and in holding that:

> "I like to think that the moon is there even if I am not looking at it".

[24] A putative model for such behaviour is given in [Wl03], §1.5, p.5: "The second way to model our real world is to assume that it is deterministic. ...It would be worthwhile to explore the consequences of a deterministic world with incomplete information (since under the assumption of determinancy in the author's eyes this comes closest to real life). That is a world in which each infinite sequence is given by an algorithm, which in most cases is completely unknown. We can model such a world by introducing two players, where player I picks algorithms and hands out the computed values of these algorithms to player II, one at a time. Sometimes player I discloses (partial) information about the algorithms themselves. Player II can of course construct her or his own algorithms, but still is confronted with recursive elements of player I about which she/he has incomplete information".

4.D. Incompleteness: Arithmetical analogy

The distinction also suggests that neither classical mechanics nor neo-classical quantum mechanics could be described as 'mathematically complete' with respect to the algorithmically verifiable behaviour of the physical world.

> The analogy here is that Gödel showed in 1931[25] that any formal arithmetic is not mathematically complete with respect to the algorithmically verifiable nature and behaviour of the natural numbers[26].
>
> However it can be argued that the first-order Peano Arithmetic PA *is* complete[27] with respect to the algorithmically computable nature and behaviour of the natural numbers.

In this sense, the EPR paper may not be entirely wrong in holding that:

> "We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete."

4.E. Conjugate properties

The above also suggests that:

Thesis 4. *The nature and behaviour of two conjugate properties F_1 and F_2 of a particle P that are determined by neo-classical laws are described mathematically at any time $t(n)$ by two algorithmically verifiable, but not algorithmically computable, functions f_1 and f_2.*

> In other words, it is the very essence of the neo-classical laws determining the nature and behaviour of the particle that—at any time $t(n)$—we can only determine either $f_1(n)$ or $f_2(n)$, but not both.
>
> Hence measuring either one makes the other indeterminate as we cannot go back in time. This does not contradict the assumption that any property of an object must obey some deterministic natural law for any possible measurement that is made at any time.

4.F. Entangled particles

The above similarly suggests that:

Thesis 5. *The nature and behaviour of an entangled property of two particles P and Q is determined by neo-classical laws, and are describable mathematically at any time $t(n)$ by two algorithmically verifiable—but not algorithmically computable—functions f_1 and g_1.*

> In other words, it is the very essence of the neo-classical laws determining the nature and behaviour of the entangled properties of two particles that—at any time $t(n)$—determining the state of one immediately gives the state of the other without measurement if the properties are entangled in a known manner.
>
> This does not contradict the assumption that any property of an object must obey some deterministic natural law for any possible measurement that is made at any time. Nor does it require any information to travel from one particle to another consequent to a measurement.

4.G. Schrödinger's cat

If $[F(x)]$ is an algorithmically verifiable but not algorithmically computable Boolean function, we can take the query:

> Is $F(n) = 0$ for all natural numbers?

as corresponding to the Schrödinger question:

[25] [Go31].

[26] Which—as shown in [An12]—is the behaviour sought to be captured by the Standard interpretation of PA.

[27] And categorical: Corollary 6.2.

> Is the cat dead or alive at any given time t?

We can then argue that there is no mathematical paradox involved in Schrödinger's assertion that the cat is both dead and alive, if we take this to mean that:

> I may either assume the cat to be alive until a given time t (in the future), or assume the cat to be dead until the time t, without arriving at any logical contradiction in my existing Quantum description of nature.

In other words:

> Once we accept Quantum Theory as a valid description of nature, then there is no paradox in stating that the theory essentially cannot predict the state of the cat at any moment of future time.
>
> The inability to predict such a state does not arise out of a lack of sufficient information about the laws of the system that Quantum theory is describing, but stems from the very nature of these laws.

The mathematical analogy for the above would be:

> Once we accept that Peano Arithmetic is consistent[28] and categorical[29] then we cannot deduce from the axioms of PA whether $F(n) = 0$ for all natural numbers, or whether $F(n) = 1$ for some natural number.

5. Are *both* interpretations of PA over the structure ℕ sound?

In the rest of this paper we shall justify that the structure ℕ of Section 3. can, indeed, be used to define both an algorithmically verifiable logic that yields the standard interpretation $\boldsymbol{M}$ of PA, which is capable of expressing algorithmically verifiable functions that can be treated as mathematically representing quantum laws of nature; and an algorithmically computable logic that yields a finitary interpretation $\boldsymbol{B}$ of PA, which is capable of expressing algorithmically computable functions that can be treated as mathematically representing only the classical laws of nature.

We shall show that, from the PA-provability of $[\neg(\forall x)F(x)]$, we may only conclude under the finitary interpretation $\boldsymbol{B}$, on the basis of evidence-based reasoning, that it is not the case that $[F(n)]$ interprets as always true in $\boldsymbol{B}$ relative to T_B.

We may *not* conclude further on the basis of evidence-based reasoning that $[F(n)]$ interprets as false in $\boldsymbol{B}$ relative to T_B for some numeral $[n]$.

More precisely, we may not conclude from the PA-provability of $[\neg(\forall x)F(x)]$, on the basis of evidence-based reasoning, that the proposition $F^*(n)$ does not hold in $\boldsymbol{B}$ for some natural number n, since we shall show that PA is *not* ω-consistent.

We therefore address the question:

- Are both the interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA over the structure ℕ sound, in the sense that the PA axioms interpret as true, and the rules of inference preserve truth, relative to each of the assignments of truth values T_M and T_B respectively?

5.A. Evidence-based reasoning

We begin by noting that the two interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA over the structure ℕ can be viewed as complementary, since[30] Tarski's classic definitions permit an intelligence—whether human or mechanistic—to admit *finitary* evidence-based inductive definitions of the satisfaction and truth of the *atomic* formulas of the first-order Peano Arithmetic PA, over the domain N of the natural numbers, in the two, hitherto unsuspected and essentially different, ways noted in Section 2. above:

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[28] Theorem 6.3 establishes the 'consistency'.
[29] Corollary 6.2 establishes the 'categoricity'—which means that any two models of the Arithmetic are isomorphic.
[30] As introduced in [An12] and detailed below.

(1) In terms of *classical* algorithmic verifiabilty; and

(2) In terms of *finitary* algorithmic computability.

Since algorithmic verifiability is defined constructively (Definition 3), we note without further comment that the Church-Turing Thesis would not hold if we were to define:

Definition 5. *An arithmetical function is effectively computable if, and only if, it is algorithmically verifiable.*

Standard Church's Thesis[31]: A number-theoretic function (or relation, treated as a Boolean function) is effectively computable if, and only if, it is partial-recursive[32].

Standard Turing's Thesis[33]: A number-theoretic function (or relation, treated as a Boolean function) is effectively computable if, and only if, it is Turing-computable[34].

5.B. Reviewing Tarski's inductive assignment of truth-values under an interpretation

We now show that the two definitions correspond to two distinctly different, hitherto unsuspected, logics corresponding to two distinctly different assignments of satisfaction and truth to the *compound* formulas of PA over $\mathbb{N}$—T_M and T_B—such that:

- The PA axioms are true over $\mathbb{N}$, and
- The PA rules of inference preserve truth over $\mathbb{N}$,

under both the interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ relative to T_M and T_B respectively.

We essentially follow standard expositions[35] of Tarski's inductive definitions on the 'satisfiability' and 'truth' of the formulas of a formal language under an interpretation where:

Definition 6. *If* $[A]$ *is an atomic formula* $[A(x_1, x_2, \ldots, x_n)]$[36] *of a formal language S, then the denumerable sequence* $(a_1, a_2, \ldots)$ *in the domain* $\mathbb{D}$ *of an interpretation* $\mathcal{I}_{S(\mathbb{D})}$ *of S satisfies* $[A]$ *if, and only if:*

(i) $[A(x_1, x_2, \ldots, x_n)]$ *interprets under* $\mathcal{I}_{S(\mathbb{D})}$ *as a unique relation* $A^*(x_1, x_2, \ldots, x_n)$ *in* $\mathbb{D}$ *for any witness* $\mathcal{W}_{\mathbb{D}}$ *of* $\mathbb{D}$*;*

(ii) there is a Satisfaction Method that provides objective evidence[37] *by which any witness* $\mathcal{W}_{\mathbb{D}}$ *of* $\mathbb{D}$ *can objectively* ***define*** *for any atomic formula* $[A(x_1, x_2, \ldots, x_n)]$ *of S, and any specified denumerable sequence* $(b_1, b_2, \ldots)$ *of* $\mathbb{D}$*, whether the proposition* $A^*(b_1, b_2, \ldots, b_n)$ *holds or not in* $\mathbb{D}$*;*

(iii) $A^*(a_1, a_2, \ldots, a_n)$ *holds in* $\mathbb{D}$ *for any* $\mathcal{W}_{\mathbb{D}}$*.*

Witness: From a constructive perspective, the existence of a 'witness' as in (i) above is implicit in the usual expositions of Tarski's definitions.

Satisfaction Method: From a constructive perspective, the existence of a Satisfaction Method as in (ii) above is also implicit in the usual expositions of Tarski's definitions.

A constructive perspective: We highlight the word '***define***' in (ii) above to emphasise the constructive perspective underlying this paper; which is that the concepts of 'satisfaction' and 'truth' under an interpretation are to be explicitly viewed as objective assignments by a convention that is witness-independent. A Platonist perspective would substitute 'decide' for 'define', thus implicitly suggesting that these concepts can 'exist', in the sense of needing to be discovered by some witness-dependent means—eerily akin to a 'revelation'—if the domain $\mathbb{D}$ is $\mathbb{N}$.

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[31] *Church's (original) Thesis*: The effectively computable number-theoretic functions are the algorithmically computable number-theoretic functions [Ch36].

[32] cf. [Me64], p.227.

[33] After describing what he meant by "computable" numbers in the opening sentence of his seminal paper on Computable Numbers [Tu36], Turing immediately expressed this thesis—albeit informally—as: "...the computable numbers include all numbers which could naturally be regarded as computable".

[34] cf. [BBJ03], p.33.

[35] See Section 8., Appendix A.

[36] We use square brackets to indicate that the contents represent a symbol or a formula of a formal theory, generally assumed to be well-formed unless otherwise indicated by the context.

[37] In the sense of [Mu91].

We further define the truth values of 'satisfaction', 'truth', and 'falsity' for the compound formulas of a first-order theory S under the interpretation $\mathcal{I}_{S(\mathbb{D})}$ in terms of *only* the satisfiability of the atomic formulas of S over $\mathbb{D}$ as usual[38].

We then have that[39]:

Theorem 5.1. *(Satisfaction Theorem) If, for any interpretation $\mathcal{I}_{S(\mathbb{D})}$ of a first-order theory S, there is an objective Satisfaction Method SM for assigning truth values to the atomic formulas of S, then:*

> *(i) The Δ_0 formulas of S are decidable as either true or false (with respect to SM) over $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$;*
>
> *(ii) If the Δ_n formulas of S are decidable as either true or as false over $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$, then so are the $\Delta(n+1)$ formulas of S.*

Proof It follows from the above definitions that:

(a) If, for any specified atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of S, it is decidable by $\mathcal{W}_{\mathbb{D}}$ whether or not a sequence $(a_1, a_2, \ldots, a_n)$ of $\mathbb{D}$ satisfies $[A(x_1, x_2, \ldots, x_n)]$ in $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$ then, for any specified compound formula $[A^1(x_1, x_2, \ldots, x_n)]$ of S containing any one of the logical constants $\neg, \rightarrow, \forall$, it is decidable by $\mathcal{W}_{\mathbb{D}}$ whether or not the sequence $(a_1, a_2, \ldots, a_n)$ of $\mathbb{D}$ satisfies $[A^1(x_1, x_2, \ldots, x_n)]$ in $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$;

(b) If, for any specified compound formula $[B^n(x_1, x_2, \ldots, x_n)]$ of S containing n of the logical constants $\neg, \rightarrow, \forall$, it is decidable by $\mathcal{W}_{\mathbb{D}}$ whether or not a sequence $(a_1, a_2, \ldots, a_n)$ of $\mathbb{D}$ satisfies $[B^n(x_1, x_2, \ldots, x_n)]$ in $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$ then, for any specified compound formula $[B^{(n+1)}(x_1, x_2, \ldots, x_n)]$ of S containing $n+1$ of the logical constants $\neg, \rightarrow, \forall$, it is decidable by $\mathcal{W}_{\mathbb{D}}$ whether or not the sequence $(a_1, a_2, \ldots, a_n)$ of $\mathbb{D}$ satisfies $[B^{(n+1)}(x_1, x_2, \ldots, x_n)]$ in $\mathbb{D}$ under $\mathcal{I}_{S(\mathbb{D})}$.

The Theorem follows. □

In other words, if the atomic formulas of of S interpret under $\mathcal{I}_{S(\mathbb{D})}$ as decidable over $\mathbb{D}$ with respect to the Satisfaction Method *SM*, then the propositions of S (i.e., the Π_n and Σ_n formulas of S) also interpret as decidable over $\mathbb{D}$ with respect to *SM*.

We note in particular that:

Theorem 5.2. *A well-formed formula $[F(x)]$ of PA is decidable as true or false under Tarski's truth assignments if, and only if, $[F(x)]$ is algorithmically verifiable.*

Proof The proof follows immediately from Definitions 12 and 13 in §8., since Tarski's definitions are inductive, and a well-formed formula $[F(x)]$ of PA is decidable as true or false under the standard interpretation $\boldsymbol{M}$ of PA over $\mathbb{N}$ if, and only if, each instantiation $[F(n)]$ of $[F(x)]$ is decidable in $\mathbb{N}$. □

We cannot, therefore, assume that the satisfaction and truth of the compound formulas of PA are always finitarily decidable—in the sense of being algorithmically computable—under the standard interpretation $\boldsymbol{M}$ of PA over $\mathbb{N}$, since we cannot prove finitarily from only Tarski's definitions and the assignment T_M of algorithmically verifiable truth values to the atomic formulas of PA under $\boldsymbol{M}$ whether, or not, a quantified PA formula $[(\forall x_i)R]$ is algorithmically verifiable as true under $\boldsymbol{M}$.

We now show how Tarski's definitions yield two distinctly different interpretations of the first-order Peano Arithmetic PA over the domain $\mathbb{N}$ of the natural numbers.

5.C. The ambiguity in the classical standard interpretation of PA over the domain $\mathbb{N}$ of the natural numbers

We note that the classical standard interpretation $\boldsymbol{M}$ of PA over the domain $\mathbb{N}$ of the natural numbers is obtained if, in $\mathcal{I}_{S(\mathbb{D})}$:

> (a) we define S as PA with standard first-order predicate calculus as the underlying logic[40];

[38] See §8., Appendix A.

[39] cf. [Me64], pp.51-53.

[40] Where the string $[(\exists \ldots)]$ is defined as—and is to be treated as an abbreviation for—the PA formula $[\neg(\forall \ldots)\neg]$. We do not consider the case where the underlying logic is Hilbert's formalisation of Aristotle's logic of predicates in terms of his ϵ-operator ([Hi27], pp.465-466).

(b) We define $\mathbb{D}$ as the set $\mathbb{N}$ of natural numbers;

(c) We assume for any atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of PA, and any specified sequence $(b_1^*, b_2^*, \ldots, b_n^*)$ of $\mathbb{N}$, that the proposition $A^*(b_1^*, b_2^*, \ldots, b_n^*)$ is decidable in $\mathbb{N}$;

(d) We define the witness $\mathcal{W}_{\mathbb{N}}$ informally as the 'mathematical intuition' of a human intelligence for whom, classically, (c) has been *implicitly* accepted as *objectively* 'decidable' in $\mathbb{N}$.

(e) We postulate that Aristotle's particularisation holds over $\mathbb{N}$[41].

Clearly, (e) does not form any part of Tarski's *inductive* definitions of the satisfaction, and truth, of the formulas of PA under the above interpretation. Moreover, its inclusion makes $\boldsymbol{M}$ extraneously non-finitary[42].

Moreover, the implicit acceptance in (d) conceals an ambiguity that needs to be made explicit since:

Lemma 5.3. *$A^*(x_1, x_2, \ldots, x_n)$ is both algorithmically verifiable and algorithmically computable in $\mathbb{N}$ by $\mathcal{W}_{\mathbb{N}}$.*

Proof (i) It follows from the argument in Theorem 5.4 (below) that $A^*(x_1, x_2, \ldots, x_n)$ is algorithmically verifiable in $\mathbb{N}$ by $\mathcal{W}_{\mathbb{N}}$.

(ii) It follows from the argument in Theorem 5.11 (below) that $A^*(x_1, x_2, \ldots, x_n)$ is algorithmically computable in $\mathbb{N}$ by $\mathcal{W}_{\mathbb{N}}$. The lemma follows. □

We note without proof that[43] (i) is consistent with, whilst (ii) is inconsistent with, the assumption of Aristotle's particularisation.

5.D. The standard interpretation $\boldsymbol{M}$ of PA

We now argue that:

Definition 7. *An atomic formula $[A]$ of PA is satisfiable under the interpretation $\boldsymbol{M}$ if, and only if, $[A]$ is algorithmically verifiable under $\boldsymbol{M}$.*

We note that:

Theorem 5.4. *The atomic formulas of PA are algorithmically verifiable as true or false under the standard interpretation $\boldsymbol{M}$.*

Proof It follows from Gödel's definition of the primitive recursive relation xBy[44]—where x is the Gödel number of a proof sequence in PA whose last term is the PA formula with Gödel-number y—that, if $[A(x_1, x_2, \ldots, x_n)]$ is an atomic formula of PA, we can algorithmically verify which one of the instantiations $[A(a_1, a_2, \ldots, a_n)]$ and $[\neg A(a_1, a_2, \ldots, x_a)]$ is necessarily PA-provable and, ipso facto, true under $\boldsymbol{M}$. □

We note that the interpretation $\boldsymbol{M}$ cannot claim to be finitary[45].

Reason: It follows from Theorem 3.1 that we cannot conclude finitarily from Tarski's definitions[46] whether or not a quantified PA formula $[(\forall x_i)R]$ is algorithmically verifiable as true under $\boldsymbol{M}$ if $[R]$ is algorithmically verifiable but not algorithmically computable under the interpretation[47].

[41] This postulates that a PA formula such as $[(\exists x)F(x)]$ can always be taken to interpret under $\boldsymbol{M}$ as 'There is some natural number n such that $F(n)$ holds in $\mathbb{N}$.

[42] As argued by Brouwer in [Br08]; see also [An15].

[43] For a more detailed argument see [An12].

[44] [Go31], p. 22(45).

[45] See [An12] for a proof that $\boldsymbol{M}$ is non-finitary, since it defines a model of PA if, and only if, PA is ω-consistent and so we may always non-finitarily conclude from $[(\exists x)R(x)]$ the existence of some numeral $[n]$ such that $[R(n)]$.

[46] Definition 6 in §5.B., and Definitions 9 to 13 in §8..

[47] Although a proof that such a PA formula exists is not obvious, we shall show that Gödel's 'undecidable' arithmetical formula $[R(x)]$ is algorithmically verifiable but not algorithmically computable under the interpretation $\boldsymbol{M}$.

5.E. The PA axioms are algorithmically verifiable as true under M

The significance of defining satisfaction in terms of algorithmic verifiability under M is that:

Lemma 5.5. *The PA axioms PA_1 to PA_8*[48] *are algorithmically verifiable as true over ℕ under the interpretation M.*

Proof Since $[x + y]$, $[x \star y]$, $[x = y]$, $[x']$ are defined recursively[49], the PA axioms PA_1 to PA_8[50] interpret as recursive relations that do not involve any quantification. The lemma follows straightforwardly from Theorem 5.4 and Tarski's definitions[51]. □

Lemma 5.6. *For any specified PA formula $[F(x)]$, the Induction axiom schema $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically verifiable true formula under M.*

Proof

(a) If $[F(0)]$ interprets as an algorithmically verifiable false formula under M the lemma is proved.

> *Reason*: Since $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically verifiable true formula under M if, and only if, either $[F(0)]$ interprets as an algorithmically verifiable false formula or $[((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)]$ interprets as an algorithmically verifiable true formula under under M.

(b) If $[F(0)]$ interprets as an algorithmically verifiable true formula, and $[(\forall x)(F(x) \rightarrow F(x'))]$ interprets as an algorithmically verifiable false formula, under M, the lemma is proved.

(c) If $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically verifiable true formulas under M then, for any natural number n, there is an algorithm which (by Definition 3) will evidence that $[F(n) \rightarrow F(n')]$ is an algorithmically verifiable true formula under M.

(d) Since $[F(0)]$ interprets as an algorithmically verifiable true formula under M, it follows for any natural number n that there is an algorithm which will evidence that each of the formulas in the finite sequence $\{[F(0),\ F(1),\ \ldots,\ F(n)\}]$ is an algorithmically verifiable true formula under the interpretation.

(e) Hence $[(\forall x)F(x)]$ is an algorithmically verifiable true formula under M.

Since the above cases are exhaustive, the lemma follows. □

> We note that if $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically verifiable true formulas under M, then we can only conclude that, for any natural number n, there is an algorithm which will give evidence for any $m \leq n$ that the formula $[F(m)]$ is true under M.
>
> We cannot conclude that there is an algorithm which, for any natural number n, will give evidence that the formula $[F(n)]$ is true under M.

Lemma 5.7. *Generalisation preserves algorithmically verifiable truth under M.*

Proof The two meta-assertions:

'$[F(x)]$ interprets as an algorithmically verifiable true formula under M[52]'

and

'$[(\forall x)F(x)]$ interprets as an algorithmically verifiable true formula under M'

both mean:

[48] As detailed in §8..
[49] cf. [Go31], p.17.
[50] As detailed in §8..
[51] Definition 6 in §5.B., and Definitions 9 to 13 in §8..
[52] See Definition 3

$[F(x)]$ is algorithmically verifiable as true under $\boldsymbol{M}$. □

It is also straightforward to see that:

Lemma 5.8. *Modus Ponens preserves algorithmically verifiable truth under* $\boldsymbol{M}$. □

We thus have that:

Theorem 5.9. *The axioms of PA are algorithmically verifiable as true under the interpretation* $\boldsymbol{M}$*, and the rules of inference of PA preserve the properties of algorithmically verifiable satisfaction/truth under* $\boldsymbol{M}$. □

By Theorem 5.4 we conclude that:

Theorem 5.10. *If the the PA-theorems interpret as algorithmically verifiable truths under* $\boldsymbol{M}$*, then PA is consistent.* □

We note that, like Gentzen's argument, such a proof of consistency would be debatably 'finitary', since we cannot conclude from Theorem 5.4 that the quantified formulas of PA are 'finitarily' decidable as true or false under the interpretation $\boldsymbol{M}$.

5.F. The finitary interpretation $\boldsymbol{B}$ of PA

We next consider a finitary interpretation $\boldsymbol{B}$ of PA, under which we define:

Definition 8. *An atomic formula* $[A]$ *of PA is satisfiable under the interpretation* $\boldsymbol{B}$ *if, and only if,* $[A]$ *is algorithmically computable under* $\boldsymbol{B}$.

We note that:

Theorem 5.11. *The atomic formulas of PA are algorithmically computable as true or as false under the finitary interpretation* $\boldsymbol{B}$.

Proof If $[A(x_1, x_2, \ldots, x_n)]$ is an atomic formula of PA then, for any specified sequence of numerals $[b_1, b_2, \ldots, b_n]$, the PA formula $[A(b_1, b_2, \ldots, b_n)]$ is an atomic formula of the form $[c = d]$, where $[c]$ and $[d]$ are atomic PA formulas that denote PA numerals. Since $[c]$ and $[d]$ are recursively defined formulas in the language of PA, it follows from a standard result[53] that $[c = d]$ is algorithmically computable as either true or false in $\mathbb{N}$ since there is an algorithm that, for any specified sequence of numerals $[b_1, b_2, \ldots, b_n]$, will give evidence whether $[A(b_1, b_2, \ldots, b_n)]$ interprets as true or false in $\mathbb{N}$. The lemma follows. □

We note that the interpretation $\boldsymbol{B}$ is finitary since:

Lemma 5.12. *The formulas of PA are algorithmically computable finitarily as true or as false under* $\boldsymbol{B}$.

Proof The Lemma follows by finite induction from Definition 4, Tarski's definitions[54], and Theorem 5.11. □

5.G. The PA axioms are algorithmically computable as true under $\boldsymbol{B}$

The significance of defining satisfaction in terms of algorithmic computability under $\boldsymbol{B}$ as above is that:

Lemma 5.13. *The PA axioms* PA_1 *to* PA_8[55] *are algorithmically computable as true under the interpretation* $\boldsymbol{B}$.

Proof Since $[x + y]$, $[x \star y]$, $[x = y]$, $[x']$ are defined recursively[56], the PA axioms PA_1 to PA_8[57] interpret as recursive relations that do not involve any quantification. The lemma follows straightforwardly from Tarski's definitions[58] and Theorem 5.4. □

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[53] For any natural numbers m, n, if $m \neq n$, then PA proves $[\neg(m = n)]$ ([Me64], p.110, Proposition 3.6). The converse is obviously true.

[54] Definition 6 in §5.B., and Definitions 9 to 13 in §8..

[55] As detailed in §8..

[56] cf. [Go31], p.17.

[57] As detailed in §8..

[58] Definition 6 in §5.B., and Definitions 9 to 13 in §8..

Lemma 5.14. *For any specified PA formula $[F(x)]$, the Induction axiom schema $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically computable true formula under* $\boldsymbol{B}$.

Proof By Tarski's definitions[59]:

(a) If $[F(0)]$ interprets as an algorithmically computable false formula under $\boldsymbol{B}$ the lemma is proved.

> Since $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically computable true formula if, and only if, either $[F(0)]$ interprets as an algorithmically computable false formula, or $[((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)]$ interprets as an algorithmically computable true formula, under $\boldsymbol{B}$.

(b) If $[F(0)]$ interprets as an algorithmically computable true formula, and $[(\forall x)(F(x) \rightarrow F(x'))]$ interprets as an algorithmically computable false formula, under $\boldsymbol{B}$, the lemma is proved.

(c) If $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically computable true formulas under $\boldsymbol{B}$, then by Definition 4 there is an algorithm which, for any natural number n, will give evidence that the formula $[F(n) \rightarrow F(n')]$ is an algorithmically computable true formula under $\boldsymbol{B}$.

(d) Since $[F(0)]$ interprets as an algorithmically computable true formula under $\boldsymbol{B}$, it follows that there is an algorithm which, for any natural number n, will give evidence that $[F(n)]$ is an algorithmically computable true formula under the interpretation.

(e) Hence $[(\forall x)F(x)]$ is an algorithmically computable true formula under $\boldsymbol{B}$.

Since the above cases are exhaustive, the lemma follows. □

Lemma 5.15. *Generalisation preserves algorithmically computable truth under* $\boldsymbol{B}$.

Proof The two meta-assertions:

'$[F(x)]$ interprets as an algorithmically computable true formula under $\boldsymbol{B}$[60]'

and

'$[(\forall x)F(x)]$ interprets as an algorithmically computable true formula under $\boldsymbol{B}$'

both mean:

$[F(x)]$ is algorithmically computable as true under $\boldsymbol{M}$. □

It is also straightforward to see that:

Lemma 5.16. *Modus Ponens preserves algorithmically computable truth under* $\boldsymbol{B}$. □

We thus have that[61]:

Theorem 5.17. *The axioms of PA are algorithmically computable as true under the interpretation $\boldsymbol{B}$, and the rules of inference of PA preserve the properties of algorithmically computable satisfaction/truth under* $\boldsymbol{B}$. □

We now show that a PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$. Hence the formulas of PA are finitarily decidable as true/false; whence PA is consistent.

[59] Definition 6 in §5.B., and Definitions 9 to 13 in §8..
[60] See Definition 4
[61] Without appeal, moreover, to Aristotle's particularisation.

6. Bridging PA Provability and Computability

We note that PA is 'computably' complete in the sense of Hilbert's ω-rule[62], so that:

Theorem 6.1. *(Provability Theorem for PA) A PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$.*

Proof We have by definition that $[(\forall x)F(x)]$ interprets as true under the interpretation $\boldsymbol{B}$ if, and only if, $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$.

By Theorem 5.17, $\boldsymbol{B}$ defines a finitary model of PA over $\mathbb{N}$ such that:

If $[(\forall x)F(x)]$ is PA-provable, then $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$;

If $[\neg(\forall x)F(x)]$ is PA-provable, then it is not the case that $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$.

Now, we cannot have that both $[(\forall x)F(x)]$ and $[\neg(\forall x)F(x)]$ are PA-unprovable for some PA formula $[F(x)]$, as this would yield the contradiction:

(i) There is a finitary model—say $\boldsymbol{B}'$—of PA+$[(\forall x)F(x)]$ in which $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$;

(ii) There is a finitary model—say $\boldsymbol{B}''$—of PA+$[\neg(\forall x)F(x)]$ in which it is not the case that $[F(x)]$ is algorithmically computable as true in $\mathbb{N}$.

The lemma follows. □

Corollary 6.2. *PA is categorical with respect to algorithmic computability.*

Since PA-provability is finitary, the assignment T_B of algorithmically computable truth values to the formulas of PA under $\boldsymbol{B}$ is finitarily decidable.

Hence the PA-theorems interpret as finitary truths under $\boldsymbol{B}$, and we have a finitary proof that:

Theorem 6.3. *PA is consistent.* □

6.A. PA is *not* ω-consistent

We further conclude that PA is *not* ω-consistent and that, since Hilbert's ε-calculus admits ε-terms that interpret as 'unspecified' natural numbers[63], the calculus—contrary to conventional wisdom[64]—is not a conservative extension of PA.

We first note that:

Lemma 6.4. *If $\boldsymbol{M}$ defines the standard model of PA over $\mathbb{N}$, then there is a PA formula $[F]$ which is algorithmically verifiable as true over $\mathbb{N}$ under $\boldsymbol{M}$ even though $[F]$ is not PA-provable.*

Proof Gödel has shown how to construct an arithmetical formula with a single variable—say $[R(x)]$[65]—such that $[R(x)]$ is not PA-provable[66], but $[R(n)]$ is instantiationally PA-provable for any specified PA numeral $[n]$. Hence, for any specified numeral $[n]$, Gödel's primitive recursive relation $xB\lceil[R(n)]\rceil$ must hold for some x. The lemma follows. □

By the argument in Theorem 6.1 it follows that:

Corollary 6.5. *The PA formula $[\neg(\forall x)R(x)]$ defined in Lemma 6.4 is PA-provable.* □

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[62] **Hilbert's ω-Rule**: If it is proved that the P-formula $[F(x)]$ interprets as a true numerical formula for each specified P-numeral $[x]$, then the P-formula $[(\forall x)F(x)]$ may be admitted as an initial formula (*axiom*) in P. cf. [Hi30], pp.485-494.

[63] See [An15].

[64] See, for instance, [Sl15].

[65] Gödel refers to this formula only by its Gödel number r ([Go31], p.25(12)).

[66] Gödel's aim in [Go31] was to show that $[(\forall x)R(x)]$ is not P-provable; by Generalisation it follows, however, that $[R(x)]$ is also not P-provable.

Corollary 6.6. *In any model of PA, Gödel's arithmetical formula $[R(x)]$ interprets as an algorithmically verifiable, but not algorithmically computable, function over $\mathbb{N}$.*

Proof Gödel has shown that $[R(x)]$[67] always interprets as an algorithmically verifiable function over $\mathbb{N}$[68]. By Corollary 6.5 $[R(x)]$ is not algorithmically computable as true in $\mathbb{N}$. □

Corollary 6.7. *PA is not ω-consistent.*[69]

Proof Gödel has shown that if PA is consistent, then $[R(n)]$ is PA-provable for any specified PA numeral $[n]$[70]. By Corollary 6.5 and the definition of ω-consistency, if PA is consistent then it is *not* ω-consistent. □

Theorem 6.8. *A PA formula can denote only algorithmically computable constants.*

Proof Corollary 6.5 implies, under the standard interpretation $\boldsymbol{M}$ of PA, that there is an 'unspecified' natural number q[71] for which the sentence $R^*(q)$ is false.

We thus conclude from Corollary 6.6 that the PA numeral corresponding to such a natural number q is not explicitly definable, by any PA formula, as a first-order term of PA which can be individually denoted within a PA formula. □

In other words, a natural number such as q above is an algorithmically uncomputable number, as in the case of the Gödel number of Turing's Halting function.

Whence, it follows from Gödel's reasoning that the PA-numeral corresponding to q is not explicitly definable, by any PA formula, as a first-order term of PA which can be individually denoted within a PA formula[72]; even though q is in the domain of the natural numbers that is defined completely by the semantics of Dedekind's second order Peano Postulates[73].

An immediate consequence of this is that Rosser's argument[74] cannot appeal to the introduction of an 'unspecified' PA-numeral—as an instantiation of an existential formula—into a PA-proof sequence by implicitly appealing[75] to the strategem of Rosser's Rule C[76], for concluding the existence of an 'undecidable' Rosser proposition[77] in an arithmetic such as PA.

Moreover, contrary to conventional wisdom:

Corollary 6.9. *Hilbert's ε-calculus is not a conservative extension of PA.*

Proof If Hilbert's ε-calculus were a conservative extension of PA, then it would be consistent and admit Rosser's proof[78] that the 'Rosser' formula—which is expressed in the language of PA and contains an existential quantifier—is undecidable in the ε-calculus[79]. However, by Theorem 6.1, there are no undecidable PA formulas. The corollary follows. □.

Corollary 6.10. *The standard interpretation $\boldsymbol{M}$ of PA does not define a finitary model of PA*[80].

Proof If PA is consistent but not ω-consistent, then Aristotle's particularisation does not hold over $\mathbb{N}$. Since the 'standard' interpretation $\boldsymbol{M}$ of PA appeals to Aristotle's particularisation, the lemma follows. □

[67] Gödel refers to this formula only by its Gödel number r; [Go31], p.25, eqn.12.

[68] [Go31], p.26(2): "$(n)\neg(nB_\kappa(17Gen\ r))$ holds"

[69] This conclusion is contrary to accepted dogma, since ω-consistency (or an equivalent such as Rosser's Rule C) is necessary for concluding the existence of 'undecidable' arithmetical propositions. See, for instance, Davis' remarks in [Da82], p.129(iii) that "...there is no equivocation. Either an adequate arithmetical logic is ω-inconsistent (in which case it is possible to prove false statements within it) or it has an unsolvable decision problem and is subject to the limitations of Gödel's incompleteness theorem".

[70] [Go31], p.26(2).

[71] If we accept the thesis in this related work-in-progress—[An14]—that there can be no non-standard numbers in any model of PA.

[72] See also [Sl15] for a similar, albeit independent, conclusion, based on considerations that can be viewed as a philosophical interpretation of Theorem 6.8.

[73] [AR02a], p.7, Dedekind's Theorems 132 and 133, and p.3, Definition 3.

[74] [Ro36].

[75] See, for instance, [Me64], p.143, Proposition 3.32.

[76] See [Me64], p.73, §7, Rule C.

[77] Which contains an existentially quantified formula.

[78] [Ro36].

[79] See, for instance, [Me64], p.143, Proposition 3.32.

[80] We note that finitists of all hues—ranging from Brouwer [Br08], to Wittgenstein [Wi78], to Alexander Yessenin-Volpin [He04]—have persistently questioned the assumption that the 'standard' interpretation $\boldsymbol{M}$ can be treated as well-defining a finitary model of PA; see also [Brm07].

7. Why the two interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA are complementary

The two interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA can thus be viewed as complementary since:

> (a) If we assume the satisfaction and truth values of the *compound* formulas of PA are always *non-finitarily* decidable under $\boldsymbol{M}$, then this assignment corresponds to the classical *non-finitary* standard interpretation $\boldsymbol{M}$ of PA over the domain $\mathbb{N}$ relative to the truth assignments T_M; from which we may further *non-finitarily* conclude on the basis of Gerhard Gentzen's transfinite reasoning that PA is consistent;

whilst:

> (b) The satisfaction and truth values of the *compound* formulas of PA are always *finitarily* decidable under the assignment $\boldsymbol{B}$, which corresponds to the *finitary* interpretation $\boldsymbol{B}$ of PA over the domain $\mathbb{N}$ relative to the truth assignments T_B; from which we may further *finitarily* conclude on the basis of evidence-based reasoning that PA is consistent.

We further note that, from such a perspective, the appropriate inference to be drawn from Gödel's 1931 paper[81] is no longer that there exist formally undecidable PA formulas—since PA is *not* ω-consistent—but that we can define PA formulas which, under interpretation, are algorithmically verifiable as true over $\mathbb{N}$, but not algorithmically computable as true over $\mathbb{N}$.

7.A. Why the two interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ of PA are *relatively* sound

We conclude that the two interpretations $\boldsymbol{M}$ and $\boldsymbol{B}$ yield models of the first-order Peano Arithmetic PA—over the structure $\mathbb{N}$ of the natural numbers—that are complementary, not contradictory.

The former yields the standard interpretation $\boldsymbol{M}$ of PA over $\mathbb{N}$, which is sound relative to the assignment T_M of algorithmically verifiable Tarskian truth values to the compound formulas of PA under $\boldsymbol{M}$[82], and which circumscribes the ambit of *non-finitary* human reasoning about 'true' arithmetical propositions.

The latter yields a finitary interpretation $\boldsymbol{B}$ of PA over $\mathbb{N}$, which is sound relative to the assignment T_B of algorithmically computable Tarskian truth values to the compound formulas of PA under $\boldsymbol{B}$[83], and which circumscribes the ambit of *finitary* mechanistic reasoning about 'true' arithmetical propositions.

The complementarity can also be expressed as the thesis that:

Thesis 6. *There can be no mechanist model of human reasoning if the standard interpretation $\boldsymbol{M}$ of PA can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions, and the finitary interpretation $\boldsymbol{B}$ of PA can be treated as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical propositions.*

Argument: Gödel has shown how to construct an arithmetical formula with a single variable—say $[R(x)]$[84]—such that $[R(x)]$ is not PA-provable, but $[R(n)]$ is instantiationally PA-provable for any specified PA numeral $[n]$. Hence, for any specified numeral $[n]$, Gödel's primitive recursive relation $xB\lceil[R(n)]\rceil$[85] must hold for some natural number m.

If we assume that any mechanical witness can only reason *finitarily* then although, for any specified numeral $[n]$, a mechanical witness can give evidence under the finitary interpretation $\boldsymbol{B}$ that the PA formula $[R(n)]$ holds in $\mathbb{N}$, no mechanical witness can conclude *finitarily* under the finitary interpretation $\boldsymbol{B}$ of PA that, for any specified numeral $[n]$, the PA formula $[R(n)]$ holds in $\mathbb{N}$.

However, if we assume that a human witness can also reason *non-finitarily*, then a human witness *can* conclude under the non-finitary standard interpretation $\boldsymbol{M}$ of PA that, for any specified numeral $[n]$, the PA formula $[R(n)]$ holds in $\mathbb{N}$.

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[81] [Go31].

[82] Theorem 5.10 in §5.E.. The soundness follows from Gerhard Gentzen's *non-finitary* proof of consistency for Arithmetic.

[83] Theorem 5.17 in §5.G.. The soundness follows from the *finitary* proof of consistency for PA detailed therein.

[84] Gödel refers to this formula only by its Gödel number r ([Go31], p.25(12)).

[85] Where xBy denotes Gödel's primitive recursive relation 'x is the Gödel-number of a proof sequence in PA whose last term is the PA formula with Gödel-number y' ([Go31], p. 22(45)); and $\lceil[R(n)]\rceil$ denotes the Gödel-number of the PA formula $[R(n)]$.

7.B. Emergence in a Mechanical Intelligence

The question arises:

Can a mechanical intelligence synthesise logic?

An interesting answer emerges if we accept that a logic of a language can be precisely defined (Definition 1) as a finite set of rules which constructively assign unique truth values:

(a) Of provability/unprovability to the formulas of the language; and

(b) Of truth/falsity to the sentences of any theory of the language that is defined semantically by an interpretation of the language over a structure.

It would then follow that, if we are given a first-order language and a structure, and we take synthesising a logic of the structure to mean identifying both some finite set of rules as above and an interpretation under which (a) and (b) hold, then such synthesis should be within the ambit of the reasoning ability of a Turing machine based mechanical intelligence.

In particular, it would then follow from Theorem 6.1 that any such mechanical intelligence can prove the PA formula:

$[(\forall x)\neg(\forall y)x > y]$,

which a human-like intelligence would interpret as the algorithmically computable true assertion that there is no largest computable natural number.

Now, if we take this assertion as corresponding to cognition of a concept of infinity, and if we consider such cognition as a sign of emergence in an intelligence, the above perspective suggests that:

Thesis 7. *The concept of infinity is an emergent feature of any Turing machine based mechanical intelligence founded on the first-order Peano Arithmetic.*

However, we note that, whereas a human-like intelligence could conceive of algorithmically verifiable, but not algorithmically computable, functions and relations that would admit the EPR phenomena without violating relativistic constraints, such conception is not possible by the logical constraints of a Turing machine based mechanical intelligence whose logic is circumscribed by the Provability Theorem 6.1 of the first-order Peano Arithmetic. Such a mechanical intelligence would, perforce, have to accept the existence of non-locality as indicating the existence of a physical phenomena that is not subject to relativistic constraints.

We conclude by noting that, since all observations of quantum phenomena depend upon mechanical artefacts whose logic is limited by Theorem 6.1 to that of a Turing machine, this suggests that:

Thesis 8. *Since the reasoning underlying the formulations, and interpretations, of the verifiable laws of quantum physics based upon the observations of mechanical artefacts is in terms of functions and (Boolean) relations that are algorithmically computable as true or false, discovery and formulation of the laws of quantum physics lies within the algorithmically computable logic and reasoning of a mechanical intelligence whose logic is circumscribed by the first-order Peano Arithmetic.*

8. Appendix A: Some comments on notations and concepts

Aristotle's particularisation: This holds that from an assertion such as:

'It is not the case that: for any specified x, $P^*(x)$ does not hold',

usually denoted symbolically by '$\neg(\forall x)\neg P^*(x)$', we may always validly infer in the classical, Aristotlean, logic of predicates[86] that:

'There exists an unspecified x such that $P^*(x)$ holds',

[86] [HA28], pp.58-59.

usually denoted symbolically by '$(\exists x)P^*(x)$'.

We note that Aristotle's particularisation implies that the classical first-order logic FOL is ω-consistent, and so we may always interpret the formal expression '$[(\exists x)\ F(x)]$' of a formal language under an interpretation as 'There exists an object s in the domain of the interpretation such that $F^*(s)$'.

We note further that Aristotle's particularisation is a *non-finitary* but fundamental tenet of classical logic unrestrictedly adopted as *intuitively obvious* by standard literature[87].

However, L. E. J. Brouwer had noted in his seminal 1908 paper on the unreliability of logical principles[88] that the commonly accepted interpretation of this formula is ambiguous if interpretation is intended over an infinite domain.

Brouwer essentially argued that:

(i) Even supposing the formula '$[P(x)]$' of a formal Arithmetical language interprets as an arithmetical relation denoted by '$P^*(x)$'; and

(ii) The formula '$[\neg(\forall x)\neg P(x)]$' interprets as the arithmetical proposition denoted by '$\neg(\forall x)\neg P^*(x)$';

(iii) The formula '$[(\exists x)P(x)]$'—which is formally defined as '$[\neg(\forall x)\neg P^*(x)]$'—need not interpret as the arithmetical proposition denoted by the usual abbreviation '$(\exists x)P^*(x)$'; and

(iv) That such postulation is invalid as a general logical principle in the absence of a means for constructing some putative object a for which the proposition $P^*(a)$ holds in the domain of the interpretation.

Hence we shall follow the convention that the assumption that '$(\exists x)P^*(x)$' is the intended interpretation of the formula '$[(\exists x)P(x)]$'—which is essentially the assumption that Aristotle's particularisation holds over the domain of the interpretation—must always be explicit.

ω-consistency: A formal system S is ω-consistent if, and only if, there is no S-formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S-provable and, second, $[F(a)]$ is S-provable for any specified S-term $[a]$.

In order to avoid intuitionistic objections to his reasoning in his seminal 1931 paper on formally undecidable arithmetical propositions[89], Gödel did not assume that the classical standard assignment $\mathcal{I}_{PA(N,\ S)}$ of PA yields a model of PA. Instead, Gödel introduced the syntactic property of ω-consistency as an explicit assumption in his formal reasoning[90]. Gödel explained at some length[91] that his reasons for introducing ω-consistency as an explicit assumption in his formal reasoning was to avoid appealing to the semantic concept of classical arithmetical truth—a concept which is implicitly based on an intuitionistically objectionable logic that assumes Aristotle's particularisation is valid over $\mathbb{N}$.

However, we note that if we assume the classical standard assignment $\mathcal{I}_{PA(N,\ S)}$ of PA yields a finitary model of PA, then PA is consistent if, and only if, it is ω-consistent. It can thus be argued that Gödel's Platonism was perhaps rooted (justifiably within the context of the implicit *non-finitary* assumption of Aristotle's particularisation in classical theory) in his implicitly held[92] *non-finitary* belief that any first-order axiomatic theory of arithmetic or set theory is ω-consistent.

Standard interpretation of PA: The classical standard interpretation $\boldsymbol{M}$ of PA over the domain $\mathbb{N}$ of the natural numbers is the one in which the logical constants have their 'usual' interpretations[93] in Aristotle's logic of predicates (which subsumes Aristotle's particularisation), and[94]:

(a) The set of non-negative integers is the domain;
(b) The symbol [0] interprets as the integer 0;

[87] See [Hi25], p.382; [HA28], p.48; [Sk28], p.515; [Go31], p.32.; [Kl52], p.169; [Ro53], p.90; [BF58], p.46; [Be59], pp.178 & 218; [Su60], p.3; [Wa63], p.314-315; [Qu63], pp.12-13; [Kn63], p.60; [Co66], p.4; [Me64], p.52(ii); [Nv64], p.92; [Li64], p.33; [Sh67], p.13; [Da82], p.xxv; [Rg87], p.xvii; [EC89], p.174; [Mu91]; [Sm92], p.18, Ex.3; [AR02b], p.94, Appendix, Rule 5(i); [BBJ03], p.102; [Cr05], p.6.

[88] [Br08].

[89] [Go31].

[90] [Go31], p.23 and p.28.

[91] In his introduction on p.9 of [Go31].

[92] [Go31], p.28.

[93] We essentially follow the definitions in [Me64], p.49.

[94] See [Me64], p.107.

(c) The symbol [$'$] interprets as the successor operation (addition of 1);
(d) The symbols [$+$] and [$\star$] interpret as ordinary addition and multiplication;
(e) The symbol [$=$] interprets as the identity relation.

The axioms of first-order Peano Arithmetic (PA)

PA$_1$ $[(x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))]$;
PA$_2$ $[(x_1 = x_2) \rightarrow (x_1' = x_2')]$;
PA$_3$ $[0 \neq x_1']$;
PA$_4$ $[(x_1' = x_2') \rightarrow (x_1 = x_2)]$;
PA$_5$ $[(x_1 + 0) = x_1]$;
PA$_6$ $[(x_1 + x_2') = (x_1 + x_2)']$;
PA$_7$ $[(x_1 \star 0) = 0]$;
PA$_8$ $[(x_1 \star x_2') = ((x_1 \star x_2) + x_1)]$;
PA$_9$ For any well-formed formula $[F(x)]$ of PA:
$[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$.

Generalisation in PA If $[A]$ is PA-provable, then so is $[(\forall x)A]$.

Modus Ponens in PA If $[A]$ and $[A \rightarrow B]$ are PA-provable, then so is $[B]$.

Hilbert's Second Problem: "When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. ... But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results. In geometry, the proof of the compatibility of the axioms can be effected by constructing a suitable field of numbers, such that analogous relations between the numbers of this field correspond to the geometrical axioms. ... On the other hand a direct method is needed for the proof of the compatibility of the arithmetical axioms."[95]

In this paper, we treat Hilbert's intent[96] behind the enunciation of his Second Problem as essentially seeking a finitary proof for the consistency of arithmetic when formalised in a language such as the first order Peano Arithmetic PA.

Tarski's inductive definitions: We shall assume that truth values of 'satisfaction', 'truth', and 'falsity' are assignable inductively to the compound formulas of a first-order theory S under the interpretation $\mathcal{I}_{S(\mathbb{D})}$ in terms of *only* the satisfiability of the atomic formulas of S over $\mathbb{D}$ as usual[97]:

Definition 9. *A denumerable sequence s of* $\mathbb{D}$ *satisfies* $[\neg A]$ *under* $\mathcal{I}_{S(\mathbb{D})}$ *if, and only if, s does not satisfy* $[A]$*;*

Definition 10. *A denumerable sequence s of* $\mathbb{D}$ *satisfies* $[A \rightarrow B]$ *under* $\mathcal{I}_{S(\mathbb{D})}$ *if, and only if, either it is not the case that s satisfies* $[A]$*, or s satisfies* $[B]$*;*

Definition 11. *A denumerable sequence s of* $\mathbb{D}$ *satisfies* $[(\forall x_i)A]$ *under* $\mathcal{I}_{S(\mathbb{D})}$ *if, and only if, specified any denumerable sequence t of* $\mathbb{D}$ *which differs from s in at most the i'th component, t satisfies* $[A]$*;*

Definition 12. *A well-formed formula* $[A]$ *of* $\mathbb{D}$ *is true under* $\mathcal{I}_{S(\mathbb{D})}$ *if, and only if, specified any denumerable sequence t of* $\mathbb{D}$*, t satisfies* $[A]$*;*

Definition 13. *A well-formed formula* $[A]$ *of* $\mathbb{D}$ *is false under* $\mathcal{I}_{S(\mathbb{D})}$ *if, and only if, it is not the case that* $[A]$ *is true under* $\mathcal{I}_{S(\mathbb{D})}$*.*

[95] Excerpted from Maby Winton Newson's English translation [Nw02] of David Hilbert's address [Hi00] at the International Congress of Mathematicians in Paris in 1900.
[96] Compare Curtis Franks' thesis in [Fr09] that Hilbert's intent behind the enunciation of his Second Problem was essentially to establish the autonomy of arithmetical truth without appeal to any debatable philosophical considerations.
[97] See [Me64], p.51; [Mu91].

9. Appendix B: The need for constructive mathematical foundations

The following perspective by Frank Waaldijk[98] emphasises the need for a universally common, constructive, foundation for the mathematical representation of elements of reality such as those considered above:

> "Our investigations lead us to consider the possibilities for 'reuniting the antipodes'. The antipodes being classical mathematics (CLASS) and intuitionism (INT). . . . It therefore seems worthwhile to explore the 'formal' common ground of classical and intuitionistic mathematics. If systematically developed, many intuitionistic results would be seen to hold classically as well, and thus offer a way to develop a strong constructive theory which is still consistent with the rest of classical mathematics. Such a constructive theory can form a conceptual framework for applied mathematics and information technology. These sciences now use an ad-hoc approach to reality since the classical framework is inadequate. . . . [*and can*] easily use the richness of ideas already present in classical mathematics, if classical mathematics were to be systematically developed along the common grounds before the unconstructive elements are brought in."[99]

> ". . . we propose that Laplacian determinism be seen in the light of constructive mathematics and Church's Thesis. This means amongst other things that infinite sequences (of natural numbers; a real number is then given by such an infinite sequence) are never 'finished', instead we see them developing in the course of time. Now a very consequent, therefore elegant interpretation of Laplacian determinism runs as follows. Suppose that there is in the real world a developing-infinite sequence of natural numbers, say α. Then how to interpret the statement that this sequence is 'uniquely determined' by the state of the world at time zero? At time zero we can have at most finite information since, according to our constructive viewpoint, infinity is never attained. So this finite information about α supposedly enables us to 'uniquely determine' α in its course of time. It is now hard to see another interpretation of this last statement, than the one given by Church's Thesis, namely that this finite information must be a (Turing-)algorithm that we can use to compute $\alpha(n)$ for any $n \in (N)$.

> With classical logic and omniscience, the previous can be stated thus: 'for every (potentially infinite) sequence of numbers $(a_n)_{n\in\mathbb{N}}$ taken from reality there is a recursive algorithm α such that $\alpha(n) = a_n$ for each $n \in \mathbb{N}$. This statement is sometimes denoted as '$\mathbf{CT}_{phys}$', . . . this classical omniscient interpretation is easily seen to fail in real life. Therefore we adopt the constructive viewpoint. The statement 'the real world is deterministic' can then best be interpreted as: 'a (potentially infinite) sequence of numbers $(a_n)_{n\in\mathbb{N}}$ taken from reality cannot be apart from every recursive algorithm α (in symbols: $\neg\forall\alpha \in \sigma_{\omega REC}\exists n \in \mathbb{N}\ [\alpha(n) \neq a_n]$)'."[100]

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