Non-heuristic approximations of prime counting functions

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Abstract. All the known approximations of $\pi(n)$ for finite values of n are derived from real-valued functions that are asymptotic to $\pi(x)$, such as $\frac{x}{log_e x}$, $Li(x)$ and Riemann's function $R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{(n)} li(x^{1/n})$. The degree of approximation for finite values of n is determined only heuristically, by conjecturing upon an error term in the asymptotic relation that can be seen to yield a closer approximation than others to the actual values of $\pi(n)$. By considering the asymptotic density of the set of all integers that are not divisible by the first $\pi(\sqrt{n})$ primes $p_1, p_2, \ldots, p_{\pi(\sqrt{n})}$ we show that, for any n, the expected number of such integers in any interval of length $(p^2_{\pi(\sqrt{n})+1} - p^2_{\pi(\sqrt{n})})$ is $(p^2_{\pi(\sqrt{n})+1} - p^2_{\pi(\sqrt{n})}) \prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$. We then show that a non-heuristic approximation—with a binomial standard deviation—for the number of primes less than or equal to n is given for all n by $\pi(n) \approx \sum_{j=1}^{n} \prod_{i=1}^{\pi(\sqrt{j})}(1 - \frac{1}{p_i}) \sim a.\frac{n}{log_e n} \rightarrow \infty$ for some constant $a > 2.e^{-\gamma} \approx 1.12292\ldots$. We further show that the expected number of Dirichlet and twin primes in the interval $(p^2_{\pi(\sqrt{n})},\ p^2_{\pi(\sqrt{n})+1})$ can be estimated similarly; and conclude that the number of such primes $\leq n$ is, in each case, cumulatively approximated non-heuristically by a function that $\rightarrow \infty$.

Keywords. Chebyshev's Theorem; complete system of incongruent residues; computational complexity; Dirichlet primes; Euler's constant γ; expected value; factorising is polynomial time; integer factorising algorithm; Mertens' theorem; mutually independent prime divisors; polynomial time algorithm; prime counting function; prime density; primes in an arithmetic progression; Prime Number Theorem; probability model; twin primes.

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Contents

1. Introduction: A 2-dimensional view of Eratosthenes sieve

> "Prime numbers are the most basic objects in mathematics. They also are among the most mysterious, for after centuries of study, the structure of the set of prime numbers is still not well understood. Describing the distribution of primes is at the heart of much mathematics . . . ".[1]

In this investigation we show how the usual, linearly displayed, Eratosthenes sieve argument reveals the structure of divisibility (and, ipso facto, of primality) more transparently when displayed as a 2-dimensional matrix representation of the residues $r_i(n)$[2], defined for all $n \geq 2$ and all $i \geq 2$ by:

$$n + r_i(n) \equiv 0\ (mod\ i), \text{ where } i > r_i(n) \geq 0.$$

Density: For instance, the residues $r_i(n)$ can be defined for all $n \geq 1$ as the values of the non-terminating sequences $R_i(n) = \{i-1,\ i-2,\ \ldots,\ 0,\ i-1,\ i-2,\ \ldots,\ 0,\ \ldots\}$, defined for all $i \geq 1$ (as illustrated below[3]).

Sequence:	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	...	R_n
$n=1$	0	1	2	3	4	5	6	7	8	9	10	...	n-1
$n=2$	0	0	1	2	3	4	5	6	7	8	9	...	n-2
$n=3$	0	1	0	1	2	3	4	5	6	7	8	...	n-3
$n=4$	0	0	2	0	1	2	3	4	5	6	7	...	n-4
$n=5$	0	1	1	3	0	1	2	3	4	5	6	...	n-5
$n=6$	0	0	0	2	4	0	1	2	3	4	5	...	n-6
$n=7$	0	1	2	1	3	5	0	1	2	3	4	...	n-7
$n=8$	0	0	1	0	2	4	6	0	1	2	3	...	n-8
$n=9$	0	1	0	3	1	3	5	7	0	1	2	...	n-9
$n=10$	0	0	2	2	0	2	4	6	8	0	1	...	n-10
$n=11$	0	1	1	1	4	1	3	5	7	9	0	...	n-11
n	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	...	0

- For any $i \geq 2$, each non-terminating sequence $R_i(n)$ cycles through the values $(i-1,\ i-2,\ \ldots,\ 0)$ with period i;
- For any $i \geq 2$ the asymptotic density[4]—over the set of natural numbers—of the set $\{n\}$ of integers that are divisible by i is $\frac{1}{i}$; and the asymptotic density of integers that are not divisible by i is $\frac{i-1}{i}$.

Primality: The residues $r_i(n)$ can alternatively be defined for all $i \geq 1$ as values of the non-terminating sequences, $E(n) = \{r_i(n) : i \geq 1\}$, defined for all $n \geq 1$ (as illustrated below[5]).

Sequence:	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	...	R_n
$E(1)$:	0	1	2	3	4	5	6	7	8	9	10	...	n-1
$E(2)$:	0	0	1	2	3	4	5	6	7	8	9	...	n-2
$E(3)$:	0	1	0	1	2	3	4	5	6	7	8	...	n-3
$E(4)$:	0	0	2	0	1	2	3	4	5	6	7	...	n-4
$E(5)$:	0	1	1	3	0	1	2	3	4	5	6	...	n-5
$E(6)$:	0	0	0	2	4	0	1	2	3	4	5	...	n-6
$E(7)$:	0	1	2	1	3	5	0	1	2	3	4	...	n-7
$E(8)$:	0	0	1	0	2	4	6	0	1	2	3	...	n-8
$E(9)$:	0	1	0	3	1	3	5	7	0	1	2	...	n-9
$E(10)$:	0	0	2	2	0	2	4	6	8	0	1	...	n-10
$E(11)$:	0	1	1	1	4	1	3	5	7	9	0	...	n-11
...													
$E(n)$:	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	...	0

- The non-terminating sequences $E(n)$ highlighted in red correspond to a prime[6] p (since $r_i(p) \neq 0$ for $1 < i < p$) in the usual, linearly displayed, Eratosthenes sieve:

$$E(\not{1}),\ E(2),\ E(3),\ E(\not{4}),\ E(5),\ E(\not{6}),\ E(7),\ E(\not{8}),\ E(\not{9}),\ E(\not{10}),\ E(11),\ \ldots$$

- The non-terminating sequences highlighted in cyan identify a crossed out composite n (since $r_i(n) = 0$ for some $1 < i < n$) in the usual, linearly displayed, Eratosthenes sieve.

[1] Andrew Granville: from this AMS press release of 5 December 1997.
[2] See §6., Appendix I(A), Fig.7 and II(B), Fig.8.
[3] For r_i read $r_i(n)$; for R_i read $R_i(n)$ in the following tables. See also Fig.7.
[4] See §1.A.(a); see also [St02], Chapter 2, p.10; [El79a], Notation, p.xxi; [GS97], Chapter 5, pp.183-186.
[5] See also Fig.8.
[6] Conventionally defined as integers that are not divisible by any smaller integer other than 1.

1.A. Thesis: The prime divisors of n are mutually independent

The residues $r_i(n)$ can thus be viewed in two essentially different ways.

> (a) *First* as the values, for any given i, of a function $R_i(n)$ over the domain N of the natural numbers. Classically, since we cannot define a probability function for the probability that a random n is prime over the probability space $(1, 2, 3, \ldots,)$, this definition does not admit an argument which will allow us to conclude that the prime divisors of any given integer n are independent.
>
> > The classical argument—that we cannot define a probability function for the probability that a random n is prime over the probability space $(1, 2, 3, \ldots,)$—is expressed informally[7] in a referee's critique of the author's original belief to the contrary:
> >
> > > "My objection is quite simply that I don't know what you mean by a randomly given positive integer n. If you want to make sense of it, then you need to assign to each positive integer n a probability $p(n)$. These probabilities must have two properties: that they are non-negative, and that their sum should be 1. If you do that, then you can talk about things like the probability that $m|n$. It will be $\sum_{d=1}^{\infty} p(dm)$.
> > >
> > > As an example, setting $p(n) = 2^{-n}$ for $n = 1, 2, 3, \ldots$ would satisfy the conditions for a probability distribution, though obviously this would be an unsuitable choice for your purposes. But the problem is that *every* possible way of choosing the $p(n)$ is unsuitable for your purposes. There does not exist a way of choosing the $p(n)$ such that for every m the equation $\sum_{d=1}^{\infty} p(dm) = 1/m$ holds.
> > >
> > > ... Consider first the probability of an unspecified integer n being divisible by an unspecified prime p. Given an arbitrary probability distribution on the positive integers, there will always be some prime p for which the above statement is false.
> > >
> > > To see this, suppose that the probability that n is chosen is not zero. Let's write this probability as $q(n)$. Now choose p so large that $1/p$ is less than $q(n)$. Then the probability that the remainder on division by p is n is at least $c(n)$ (since there is a probability $c(n)$ of choosing the integer n) and that is greater than $1/p$.
> > >
> > > ... A typical way that number theorists deal with a difficulty like this is to choose a random integer n in the range from N to $2N$ for some large integer N. But then you cannot say that the probability that n is a multiple of p is exactly $1/p$—it is only *approximately* $1/p$. And the various events are not exactly independent but only *approximately* independent. So there are error terms involved. And the entire difficulty of the subject is that these error terms accumulate and it becomes hard to say what the final answer is to any accuracy.
> > >
> > > ... Let me explain why what I did say is true. We pick an integer n uniformly at random from the set $\{N, N + 1, N + 2, , 2N\}$. What is the probability that n is even? If N is odd, then exactly half those integers are odd and half are even.
> > >
> > > If N is even, then we can write $N = 2M$, and in that case of the $N + 1$ elements of the set, $M + 1$ are even and M are odd, so the probability that n is even is $(M + 1)/2M$. So that's already an example where the probability is only approximately equal to $1/p$ (which in this case is $1/2$). In general, the

[7] See also [HL23], pp.36-37. A formal argument is given in [St02], Chapter 2, p.9, Theorem 2.1.

> number of multiples of p in a set of R consecutive integers will be R/p if p happens to be a factor of R, and otherwise it will be one of the integers on either side of R/p.
>
> In the second case, which has to happen for several p (since R cannot be divisible by every prime less than R, or even than the square root of R), the best we can say is that the probability that an integer chosen uniformly at random from the R consecutive integers is a multiple of p is approximately equal to $1/p$.
>
> ...It is possible to define a notion of "density" for sets of integers in such a way that the density of the set of all integers congruent to a mod p is $1/p$ for every a and every p.
>
> ...It is not possible to define a probability distribution on the integers in such a way that every integer is chosen with equal probability.
>
> .... If you want to claim that you can make sense of the statement:
>
> 'The probability that an unspecified integer n is divisible by p is $1/p$',
>
> you will need to develop some kind of probability theory that allows you to do something that conventional probability theory (where you would need to specify a probability distribution on the positive integers) does not."

(b) *Second* as the values, for any given n, of the sequence $E(n) = \{r_i(n) : i \geq 1\}$.

This now allows us to define a probability model from which we may conclude for any given $n > 1$, and any given prime $p > 1$, that the probability of the event $r_p(n) = 0$—whence p divides n—is $\frac{1}{p}$; and that the probability of the event $r_p(n) \neq 0$—whence p does not divide n—is $1 - \frac{1}{p}$.

This further allows us to argue (in §2.B.) that, given $p, q > 1$ are two unequal primes, the compound probability that $r_p(n) = 0$ and $r_q(n) = 0$—whence both p and q divide n—is $\frac{1}{pq}$; and so the prime divisors of any given integer n are mutually independent.

1.B. Proof: The prime divisors of n are mutually independent

(1) In this investigation accordingly—instead of addressing the probability that a *random* n is prime over the probability space $(1, 2, 3, \ldots,)$—we shall address the questions:

(a) What is the probability for any *given* $n > i > 1$ and $i \geq 0$, where $i > u \geq 0$, that:

$$n + u \equiv \ 0 \ (mod \ i)?$$

(b) What is the compound probability for any *given* $n > i, \ j > 1$, where $i \neq j$, $i > u \geq 0$, and $j > v \geq 0$, that:

$$n + u \equiv \ 0 \ (mod \ i), \text{ and } n + v \equiv \ 0 \ (mod \ j)?$$

(c) What is the compound probability for any *given* $n > i, \ j > 1$, where $i \neq j$, $i > u \geq 0$, and $j > v \geq 0$, that:

i divides $n + u$, and j divides $n + v$?

(2) We shall argue that:

(a) The answer to query (1a) above is that the probability the roll of an i-sided cylindrical die will yield the value u is $\frac{1}{i}$ by the probability model for such an event as definable over the probability space $(0, 1, 2, \ldots, i-1)$;

(b) The answer to query (1b) above is that the probability the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die will yield the values u and v, respectively, is $\frac{1}{i.j}$ by the probability model for such a simultaneous event as defined over the probability space $\{(u, v) : i > u \geq 0, j > v \geq 0\}$.

(3) We shall *trivially* conclude that:

The compound probability of determining u and v correctly from the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die, is the product of the probability of determining u correctly from the roll of an i-sided cylindrical die, and the probability of determining v correctly from the roll of a j-sided cylindrical die.

(4) We shall further conclude *non-trivially* that the answer to query (1c) above is given by:

(a) If i and j are co-prime, the compound probability of correctly determining that i divides n and j divides n from the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die, is the product of the probability of correctly determining that i divides n from the roll of an i-sided cylindrical die, and the probability of correctly determining that j divides n from the roll of a j-sided cylindrical die.

(b) The assumption that i and j be co-prime is also necessary, since 4(a) would not always be the case if i and j were not co-prime.

For instance, let $j = 2i$. The probability that an i-sided cylindrical die will then yield 0—and allow us to conclude that i divides n—is $\frac{1}{i}$, and the probability that a j-sided cylindrical die will then yield 0—and allow us to conclude that j divides n—is $\frac{1}{j}$; but the probability of determining both that i divides n, and that j divides n, from a simultaneous roll of the two cylindrical dice is $\frac{1}{j}$, and not $\frac{1}{i.j}$.

(5) We shall also conclude *non-trivially* that, if p and q are two unequal primes, the probability of algorithmically determining whether p divides n is independent of the probability of algorithmically determining whether q divides n; which yields, amongst others (see §2.C.a.), the following consequences.

1.C. Consequences: The prime divisors of n are mutually independent

By considering the asymptotic density of the set of all integers that are not divisible by the first k primes $p_1, p_2, \ldots, p_k$ we shall show that the expected number of such integers in any interval of length $(p^2_{\pi(\sqrt{n})+1} - p^2_{\pi(\sqrt{n})})$ is $\{(p^2_{\pi(\sqrt{n})+1} - p^2_{\pi(\sqrt{n})}) \prod_{i=1}^{k}(1 - \frac{1}{p_i})\}$.

This then allows us to conclude non-heuristically that:

- For each n, the expected number of primes in the interval $(p^2_{\pi(\sqrt{n})}, p^2_{\pi(\sqrt{n})+1})$ is $\{(p^2_{\pi(\sqrt{n})+1} - p^2_{\pi(\sqrt{n})}) \prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})\}$.

  The number $\pi(n)$ of primes $\leq n$ is thus cumulatively approximated (Lemma 3.8 and Corollary 3.12) for $n \geq 4$ by $\pi(n) \approx \sum_{j=1}^{n} \prod_{i=1}^{\pi(\sqrt{j})}(1 - \frac{1}{p_i}) \sim a.\frac{n}{log_e n} \rightarrow \infty$.

• For each n, the expected number of Dirichlet primes—of the form $a+m.d$ for some natural number $m \geq 1$—in the interval $(p^2_{\pi(\sqrt{n})}, p^2_{\pi(\sqrt{n})+1})$ is $\{(p^2_{\pi(\sqrt{n})+1} - p^2_{\pi(\sqrt{n})}) \prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} . \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} . \prod_{j=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_j})\}$, where $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2} \ldots q_k^{\alpha_k}$ and $(a,d) = 1$.

The number $\pi_{(a,d)}(n)$ of Dirichlet primes $\leq n$ is thus cumulatively approximated (Lemma 4.9) for all $n \geq q_k^2$ by $\pi_{(a,d)}(n) \approx \prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} . \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} . \sum_{l=1}^{n} \prod_{j=1}^{\pi(\sqrt{l})} (1 - \frac{1}{p_j}) \rightarrow \infty$.

• For each n, the expected number of $\mathbb{TW}$ primes—such that n is a prime and $n+2$ is either a prime or $p^2_{\pi(\sqrt{n})+1}$—in the interval $(p^2_{\pi(\sqrt{n})}, p^2_{\pi(\sqrt{n})+1})$ is $\{(p^2_{\pi(\sqrt{n})+1} - p^2_{\pi(\sqrt{n})}) \prod_{i=2}^{\pi(\sqrt{n})} (1 - \frac{2}{p_i})\}$.

The number $\pi_2(p^2_{k+1})$ of twin primes $\leq p^2_{k+1}$ is thus cumulatively approximated (Lemma 5.7) for all $k \geq 1$ by $\pi_2(p^2_{k+1}) \approx \sum_{j=9}^{p^2_{k+1}} \prod_{i=2}^{\pi(\sqrt{j})-1} (1 - \frac{2}{p_i}) \rightarrow \infty$.

1.D. The functions $\pi(x)$ and $\frac{x}{log_e x}$: A historical perspective

Fig.1: The asymptotic behaviour of the primes

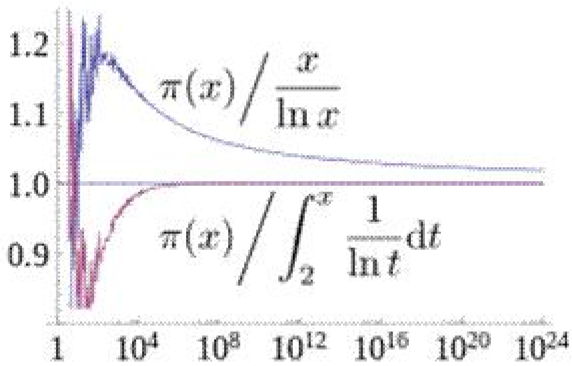

Fig.1: Graph showing ratio of the prime-counting function $\pi(x)$ to two of its approximations, $\frac{x}{ln\ x}$ and $Li(x)$. As x increases (note x axis is logarithmic), both ratios tend towards 1. The ratio for $\frac{x}{ln\ x}$ converges from above very slowly, while the ratio for $Li(x)$ converges more quickly from below.[8]

To place this investigation in an appropriate historical perspective, we note that Adrien-Marie Legendre and Carl Friedrich Gauss are reported[9] to have independently conjectured in 1796 that, if $\pi(x)$ denotes the number of primes less than x, then $\pi(x)$ is asymptotically equivalent to $\frac{x}{log_e x}$.

Around 1848/1850, Pafnuty Lvovich Chebyshev proved that $\pi(x) \asymp \frac{x}{log_e x}$, and confirmed that if $\pi(x)/\frac{x}{log_e x}$ has a limit, then it must be 1[10].

The question of whether $\pi(x)/\frac{x}{log_e x}$ has a limit at all, or whether it oscillates, was answered—it has a limit—first by Jacques Hadamard and Charles Jean de la Vallée Poussin independently in 1896, using advanced argumentation involving functions of a complex variable[11]; and again independently

[8] cf. Prime Number Theorem. (2014, June 10). In Wikipedia, The Free Encyclopedia. Retrieved 09:53, July 9, 2014, from http://en.wikipedia.org/w/index.php?titleP̄rime_number_theorem&oldid=612391868.

[9] cf. Prime Number Theorem. (2014, June 10). In Wikipedia, The Free Encyclopedia. Retrieved 09:53, July 9, 2014, from http://en.wikipedia.org/w/index.php?titleP̄rime_number_theorem&oldid=612391868; see also [Gr95].

[10] [Di52], p.439; see also [HW60], p.9, Theorem 7 and p.345, §22.4 for a proof of Chebychev's Theorem.

[11] [Di52], p.439; see also [Ti51], Chapter III, p.8 for details of Hadamard's and de la Vallée Poussin's proofs of the Prime Number Theorem.

by Paul Erdös and Atle Selberg[12] in 1949/1950, using only elementary—but still abstruse—methods without involving functions of a complex variable.

1.E. A better heuristic approximation to $\pi(x)$: The integral $Li(x)$

We also note that, reportedly[13]:

> "In a handwritten note on a reprint of his 1838 paper 'Sur l'usage des séries infinies dans la théorie des nombres', which he mailed to Carl Friedrich Gauss, Peter Gustav Lejeune Dirichlet conjectured (under a slightly different form appealing to a series rather than an integral) that an even better approximation to $\pi(x)$ is given by the offset logarithmic integral $Li(x)$ defined by:
>
> $Li(x) = \int_2^x \frac{1}{log_e t}.dt = li(x) - li(2)$."[14]

Fig.2: The distribution of the primes

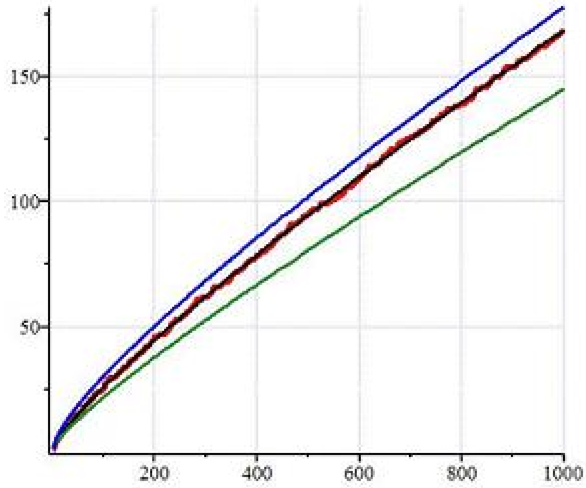

Fig.2: The above graph compares the actual number $\pi(x)$ (red) of primes $\leq x$ with the distribution of primes as estimated variously by the functions $Li(x)$ (blue), $R(x)$ (black), and $\frac{x}{log_e x}$ (green), where $R(x)$ is Riemann's function $\sum_{n=1}^{\infty} \frac{\mu(n)}{(n)} li(x^{1/n})$.[15]

We further note that in 1889 Jean de la Vallée Poussin proved[16] (cf. Fig.1):

> "...that $Li(x)$ represents $\pi(x)$ more exactly than $\frac{x}{log_e x}$ and its remaining approximations $\frac{x}{log_e x} + \frac{x}{log_e^2 x} + \ldots + \frac{(m-1)!x}{log_e^m x}$."

We note that all the known approximations of $\pi(n)$ for finite values of n are derived from real-valued functions that are only known to be asymptotic to $\pi(x)$, such as $\frac{x}{log_e x}$, $Li(x)$ and Riemann's function $R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{(n)} li(x^{1/n})$.

Consequently, the degree of approximation for finite values of n is determined only heuristically, by conjecturing upon an error term in the asymptotic relation that can be seen to yield the closest approximation upon comparison with the actual values of $\pi(n)$ within a finite range of values of n (eg. Fig.2, where $n = 1000$.).

[12] See [HW60], p.360, Theorem 433 for a proof of Selberg's Theorem.

[13] cf. Prime Number Theorem. (2014, June 10). In Wikipedia, The Free Encyclopedia. Retrieved 09:53, July 9, 2014, from: http://en.wikipedia.org/w/index.php?titlēPrime_number_theorem&oldid=612391868.

[14] Where $li(x) = \int_0^x \frac{1}{log_e t}.dt$.

[15] cf. How Many Primes Are There? In *The Prime Pages.* Retrieved 10:29, September 27, 2015, from: https://primes.utm.edu/howmany.html.

[16] [Di52], p.440.

1.F. A non-heuristic cumulative approximation of $\pi(n)$ for *all* values of n

The question arises: Is there a function which approximates $\pi(n)$ non-heuristically for all values of n?

In this investigation we shall address the above question by showing that the asymptotic density[17] of integers co-prime to the first k primes, $p_1, p_2, \ldots, p_k$, over the set of natural numbers, is:

$\prod_{i=1}^{k}(1-\frac{1}{p_i})$;

and that the expected number of such integers in the interval (a, b) is thus:

$(b-a)\prod_{i=1}^{k}(1-\frac{1}{p_i})$,

where the binomial standard deviation of the expected number of integers co-prime to $p_1, p_2, \ldots, p_k$ in any interval of length $(b-a)$ is:

$\sqrt{(b-a)\prod_{i=1}^{k}(1-\frac{1}{p_i})(1-\prod_{i=1}^{k}(1-\frac{1}{p_i}))}$.

1.G. Expected number of primes in the interval $(p^2_{\pi(\sqrt{n})},\ p^2_{\pi(\sqrt{n})+1})$

Fig.3: Comparative non-heuristically estimated distributions of the primes ≤ 3000

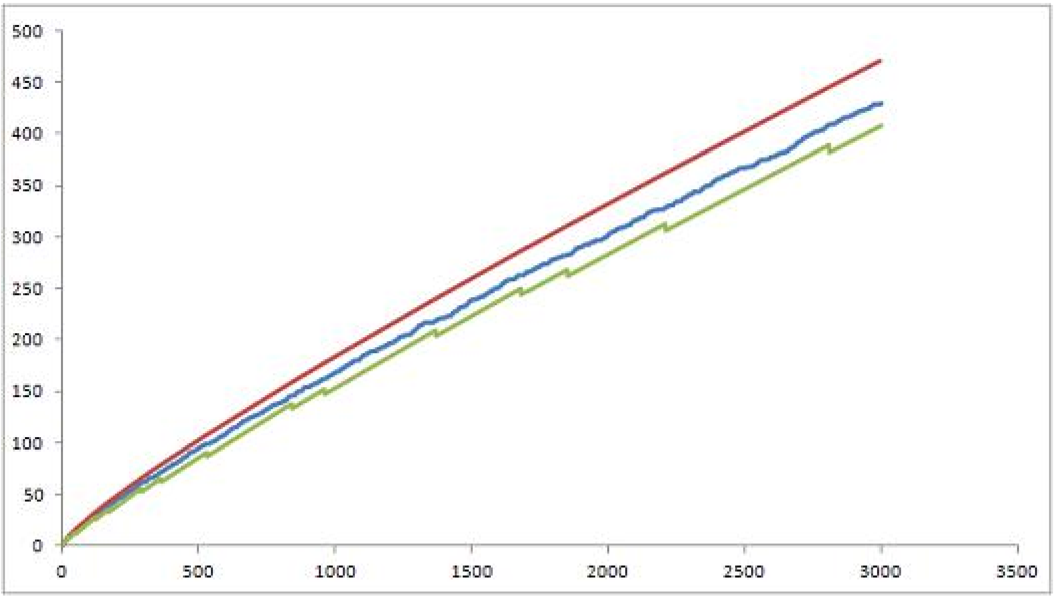

Fig.3: The above graph compares the non-heuristically estimated values of $\pi_L(n) = \sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-\frac{1}{p_i})$ (red) and $\pi_H(n) = \sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{n})}(1-\frac{1}{p_i}) = n.\prod_{i=1}^{\pi(\sqrt{n})}(1-\frac{1}{p_i})$ (green) vs the actual values of $\pi(n)$ (blue) for $4 \leq n \leq 3000$[18].

Taking (a, b) as the interval $(p^2_{\pi(\sqrt{n})},\ p^2_{\pi(\sqrt{n})+1})$, we conclude, for instance, that cumulative non-heuristic estimates of the number $\pi(p^2_{\pi(\sqrt{n})+1})$ of primes less than $p^2_{\pi(\sqrt{n})+1}$ are given by $\pi_H(n) = \sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{n})}(1-\frac{1}{p_i}) = n.\prod_{i=1}^{\pi(\sqrt{n})}(1-\frac{1}{p_i})$ (green in Fig.3) and $\pi_L(n) = \sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-\frac{1}{p_i})$ (red in Fig.3) , where:

[17] cf. [St02], Chapter 2, p.10.

[18] Query (see also Appendix II, §7.): Which is the least n such that $\pi_H(n) > \pi(n)$ (as implied by the Prime Number Theorem)?

(i) $\pi_H(p^2_{\pi(\sqrt{n})+1}) = p^2_{\pi(\sqrt{n})+1} \prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$ estimates $\pi(p^2_{\pi(\sqrt{n})+1})$ with standard deviation:

$$p_{\pi(\sqrt{n})+1}\sqrt{\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})(1 - \prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i}))}.$$

(ii) $\pi_L(p^2_{\pi(\sqrt{n})+1}) = \sum_{j=1}^{\pi(\sqrt{n})}\{(p^2_{j+1} - p^2_j)\prod_{i=1}^{j}(1 - \frac{1}{p_i})\}$ estimates $\pi(p^2_{\pi(\sqrt{n})+1})$ with cumulative standard deviation:

$$\sum_{j=1}^{\pi(\sqrt{n})}\sqrt{(p^2_{j+1} - p^2_j)\prod_{i=1}^{j}(1 - \frac{1}{p_i})(1 - \prod_{i=1}^{j}(1 - \frac{1}{p_i}))}.$$

Fig.4: Two graphs of $y = \prod_{i=1}^{\pi(\sqrt{x})}(1 - \frac{1}{p_i})$

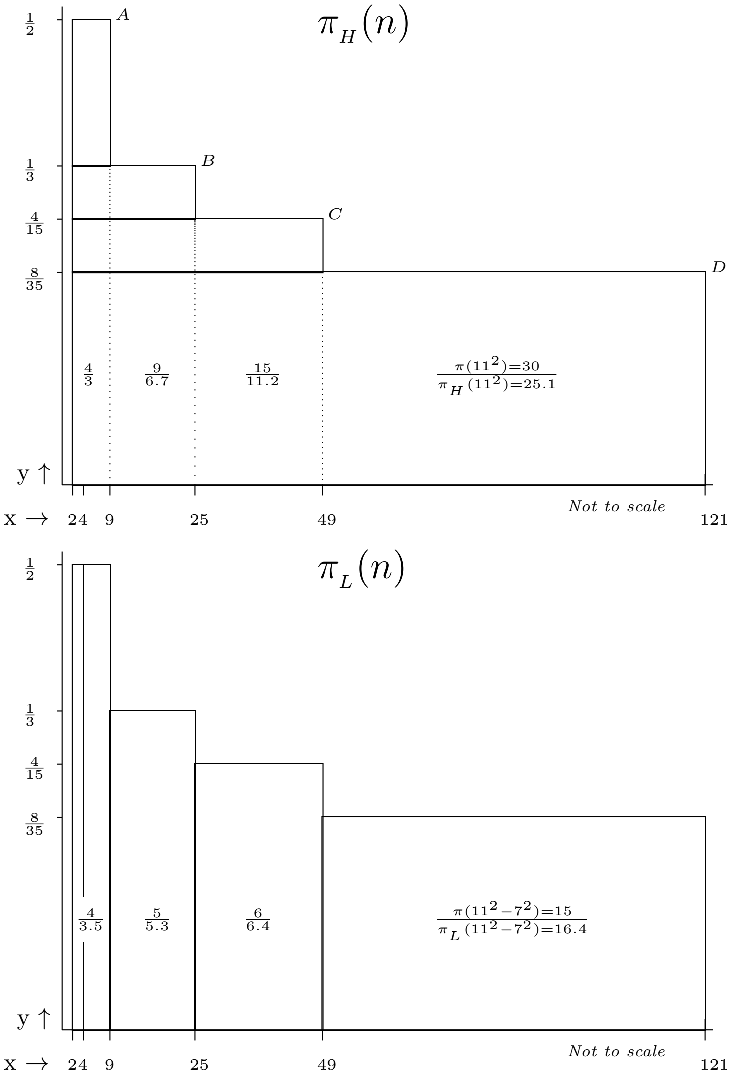

Fig.4: Two graphs of $y = \prod_{i=1}^{\pi(\sqrt{x})}(1 - \frac{1}{p_i})$. (i) The overlapping rectangles $A, B, C, D, \ldots$ in fig. $\pi_H(n)$ represent $\pi_H(p^2_{j+1}) = p^2_{j+1}.\prod_{i=1}^{j}(1 - \frac{1}{p_i})$ for $j \geq 1$. Figures within each rectangle are the primes and estimated primes corresponding to the functions $\pi(n)$ and $\pi_H(n)$, respectively, within the interval $(1,\ p^2_{j+1})$ for $j \geq 2$. (ii) The rectangles in fig. $\pi_L(n)$ represent $(p^2_{j+1} - p^2_j)\prod_{i=1}^{j}(1 - \frac{1}{p_i})$ for $j \geq 1$. Figures within each rectangle are the primes corresponding to the functions $\pi(n)$ and $\pi_L(n)$ within the interval $(p^2_j,\ p^2_{j+1})$ for $j \geq 2$. The area under the curve is $\pi_L(x) = (x - p^2_n)\prod_{i=1}^{n}(1 - \frac{1}{p_i}) + \sum_{j=1}^{n-1}(p^2_{j+1} - p^2_j)\prod_{i=1}^{j}(1 - \frac{1}{p_i}) + 2$.

(ii) and, more generally, cumulative non-heuristic approximations of the number $\pi(n)$ of primes less than or equal to n are given by the prime counting functions $\pi_L(n)$ (Lemma 3.8) and $\pi_H(n)$ (Lemma 3.5) (cf. Fig.5)[19]:

$$\pi(n) \approx \pi_H(n) = \sum_{j=1}^{n} \prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i}) = n.\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i}) \sim 2e^{-\lambda}\frac{n}{log_e n}.^{20}$$
$$\pi(n) \approx \pi_L(n) = \sum_{j=1}^{n} \prod_{i=1}^{\pi(\sqrt{j})}(1 - \frac{1}{p_i}) \sim a.\frac{n}{log_e n} \rightarrow \infty,\ a > 2.e^{-\gamma} \approx 1.12292\ldots;$$

Fig.5: The graphs of $y = \pi_H(x)$ and $y = \pi_L(x)$

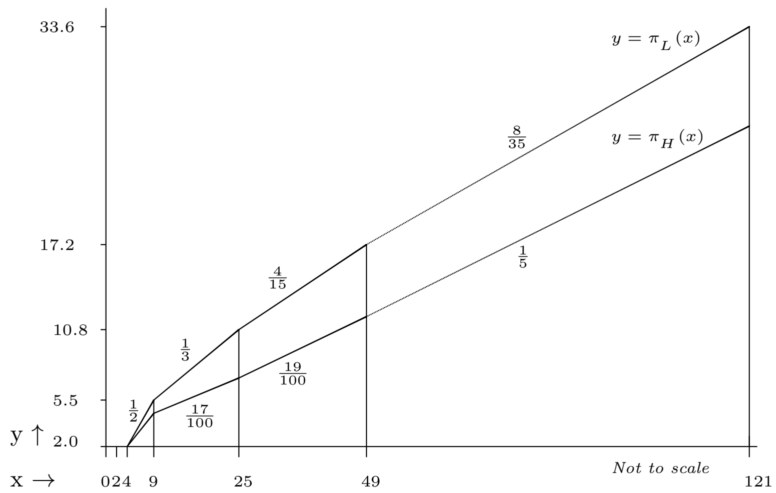

Fig.5: Graph of: (i) $y = \pi_H(x) = x.\prod_{i=1}^{\pi(\sqrt{x})}(1 - \frac{1}{p_i})$[21]; and of: (ii) $y = \pi_L(x) = (x - p_n^2)\prod_{i=1}^{n}(1 - \frac{1}{p_i}) + \sum_{j=1}^{n-1}(p_{j+1}^2 - p_j^2)\prod_{i=1}^{j}(1 - \frac{1}{p_i}) + 2$ in the interval $(p_n^2,\ p_{n+1}^2)$. Note that the gradient of $y = \pi_L(x)$ in the interval $(p_n^2,\ p_{n+1}^2)$ is $\prod_{i=1}^{n}(1 - \frac{1}{p_i}) \rightarrow 0$.

1.H. Non-heuristic approximations of prime counting functions

In the rest of this investigation we formally consider elementary, non-heuristic, arguments for:

(i) *Dirichlet's Theorem*: We show that the number $\pi_{(a,d)}(n)$ of Dirichlet primes of the form $a + m.d$ which are less than or equal to n, where a, d are co-prime and $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_k}$ (q_i prime), is non-heuristically approximated by the cumulative Dirichlet prime counting function $\pi_D(n)$ (Definition 5), such that (Lemma 4.9):

$$\pi_{(a,d)}(n) \approx \pi_D(n) = \prod_{i=1}^{k}\frac{1}{q_i^{\alpha_i}}.\prod_{i=1}^{k}(1 - \frac{1}{q_i})^{-1}.\pi_L(n) \rightarrow \infty.$$

(ii) *Twin Prime Theorem*: We show that (Theorem 5.8) there are infinitely many twin primes since a cumulative non-heuristic approximation of the number $\pi_2(p_{k+1}^2)$ of twin primes $\leq p_{k+1}^2$ for all $k \geq 1$ is:

$$\sum_{j=9}^{p_{k+1}^2}\prod_{i=2}^{\pi(\sqrt{j})-1}(1 - \frac{2}{p_i}) \rightarrow \infty.$$

[19] Compare [HL23], pp.36-37. See also §7., Appendix II for the estimated values of $\pi_L(n)$, $\pi_H(n)$, and the actual values of $\pi(n)$, for $4 \leq n \leq 1500$.

[20] Query: Which is the least n such that $\pi_H(n) > \pi(n)$ (as implied by the Prime Number Theorem)?

[21] See §3.A.

2. The residues $r_i(n)$.

We begin by formally defining the residues $r_i(n)$ for all $n \geq 2$ and all $i \geq 2$ as below[22]:

Definition 1. *$n + r_i(n) \equiv 0$ (mod i) where $i > r_i(n) \geq 0$.*

Since each residue $r_i(n)$ cycles over the i values $(i-1, i-2, \ldots, 0)$, these values are all incongruent and form a complete system of residues[23] *mod* i.

It immediately follows that:

Lemma 2.1. *$r_i(n) = 0$ if, and only if, i is a divisor of n.* □

2.A. The probability model $\mathbb{M}_i = \{(0, 1, 2, \ldots, i-1),\ r_i(n), \frac{1}{i}\}$

By the standard definition of the probability $\mathbb{P}(e)$ of an event e[24], we have by Lemma 2.1 that:

Lemma 2.2. *For any $n \geq 2$, $i \geq 2$ and any given integer $i > u \geq 0$:*

- *the probability $\mathbb{P}(r_i(n) = u)$ that $r_i(n) = u$ is $\frac{1}{i}$;*
- *$\sum_{u=0}^{u=i-1} \mathbb{P}(r_i(n) = u) = 1$;*
- *and the probability $\mathbb{P}(r_i(n) \neq u)$ that $r_i(n) \neq u$ is $1 - \frac{1}{i}$.* □

By the standard definition of a probability model, we conclude that:

Theorem 2.3. *For any $i \geq 2$, $\mathbb{M}_i = \{(0, 1, 2, \ldots, i-1),\ r_i(n), \frac{1}{i}\}$ yields a probability model for each of the values of $r_i(n)$.* □

Corollary 2.4. *For any given n, i and u such that $r_i(n) = u$, the probability that the roll of an i-sided cylindrical die will yield the value u is $\frac{1}{i}$ by the probability model defined in Theorem 2.3 over the probability space $(0, 1, 2, \ldots, i-1)$.* □

Corollary 2.5. *For any $n \geq 2$ and any prime $p \geq 2$, the probability $\mathbb{P}(r_p(n) = 0)$ that $r_p(n) = 0$, and that p divides n, is $\frac{1}{p}$; and the probability $\mathbb{P}(r_p(n) \neq 0)$ that $r_p(n) \neq 0$, and that p does not divide n, is $1 - \frac{1}{p}$.* □

We also note the standard definition[25]:

Definition 2. *Two events e_i and e_j are mutually independent for $i \neq j$ if, and only if, $\mathbb{P}(e_i \cap e_j) = \mathbb{P}(e_i).\mathbb{P}(e_j)$.*

[22] The residues $r_i(n)$ can also be graphically displayed variously as shown in the Appendix I in §6..
[23] [HW60], p.49.
[24] See [Ko56], Chapter I, §1, Axiom III, pg.2.
[25] See [Ko56], Chapter VI, §1, Definition 1, pg.57 and §2, pg.58.

2.B. The prime divisors of any integer n are mutually independent: Overview

(1) We next address the questions:

(a) What is the probability for any given $n > i > 1$ and $i \geq 0$, where $i > u \geq 0$, that:

$n + u \equiv 0 \ (mod \ i)$?

(b) What is the compound probability for any given $n > i, \ j > 1$, where $i \neq j$, $i > u \geq 0$, and $j > v \geq 0$, that:

$n + u \equiv 0 \ (mod \ i)$, and $n + v \equiv 0 \ (mod \ j)$?

(c) What is the compound probability for any given $n > i, \ j > 1$, where $i \neq j$, $i > u \geq 0$, and $j > v \geq 0$, that:

i divides $n + u$, and j divides $n + v$?

(2) We shall argue that:

(a) The answer to query (1a) above is that the probability the roll of an i-sided cylindrical die will yield the value u is $\frac{1}{i}$ by the probability model for such an event as definable over the probability space $(0, 1, 2, \ldots, i-1)$;

(b) The answer to query (1b) above is that the probability the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die will yield the values u and v, respectively, is $\frac{1}{i.j}$ by the probability model for such a simultaneous event as defined over the probability space $\{(u, v) : i > u \geq 0, j > v \geq 0\}$.

(3) We shall trivially conclude that:

The compound probability of determining u and v correctly from the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die, is the product of the probability of determining u correctly from the roll of an i-sided cylindrical die, and the probability of determining v correctly from the roll of a j-sided cylindrical die.

(4) We shall further conclude non-trivially that the answer to query (1c) above is given by:

(a) If i and j are co-prime, the compound probability of correctly determining that i divides n and j divides n from the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die, is the product of the probability of correctly determining that i divides n from the roll of an i-sided cylindrical die, and the probability of correctly determining that j divides n from the roll of a j-sided cylindrical die.

(b) The assumption that i and j be co-prime is also necessary, since 4(a) would not always be the case if i and j were not co-prime.

For instance, let $j = 2i$. The probability that an i-sided cylindrical die will then yield 0—and allow us to conclude that i divides n—is $\frac{1}{i}$, and the probability that a j-sided cylindrical die will then yield 0—and allow us to conclude that j divides n—is $\frac{1}{j}$; but the probability of determining both that i divides n, and that j divides n, from a simultaneous roll of the two cylindrical dice is $\frac{1}{j}$, and not $\frac{1}{i.j}$.

(5) We shall also conclude non-trivially that, if p and q are two unequal primes, the probability of determining whether p divides n is independent of the probability of determining whether q divides n.

2.C. The prime divisors of any integer n are mutually independent

We begin by formally noting first that:

Lemma 2.6. *If $n \geq 2$ and $n > i, j > 1$, where $i \neq j$, then:*

$$\mathbb{P}((r_i(n) = u) \cap (r_j(n) = v)) = \mathbb{P}(r_i(n) = u).\mathbb{P}(r_j(n) = v)$$

where $i > u \geq 0$ and $j > v \geq 0$.

Proof: (i) If $n \geq 2$ and $n > i, j > 1$, where $i \neq j$, then we can always determine a unique pair of residues $r_i(n) = u$ and $r_j(n) = v$, where $i > u \geq 0$, $j > v \geq 0$, i divides $n + u$, and j divides $n + v$.

(ii) There are $i.j$ pairs (u, v) such that $i > u \geq 0$ and $j > v \geq 0$.

(iii) The compound probability that the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die will yield the values u and v, respectively, is thus $\frac{1}{i.j}$ by the probability model for such a simultaneous event as defined over the probability space $\{(u, v) : i > u \geq 0, j > v \geq 0\}$, where we note that:

- the probability $\mathbb{P}((r_i(n) = u) \cap (r_j(n) = v))$ that $r_i(n) = u$ and $r_j(n) = v$ is $\frac{1}{i.j}$;
- $\sum_{All\ (u,v):\ i>u\geq 0,\ j>v\geq 0} \mathbb{P}((r_i(n) = u) \cap (r_j(n) = v)) = 1$;

(iv) By Lemma 2.2, the product of the probability $\frac{1}{i}$ that the roll of an i-sided cylindrical die will yield the value u, and the probability $\frac{1}{j}$ that the roll of a j-sided cylindrical die will yield the value v, is $\frac{1}{i.j}$.[26]

(v) It follows that:

$$\mathbb{P}((r_i(n) = u) \cap (r_j(n) = v)) = \frac{1}{i.j}$$

$$\mathbb{P}(r_i(n) = u).\mathbb{P}(r_j(n) = v) = (\tfrac{1}{i})(\tfrac{1}{j}).$$

The lemma follows. □

Corollary 2.7. $\mathbb{P}((r_i(n) = 0) \cap (r_j(n) = 0)) = \mathbb{P}(r_i(n) = 0).\mathbb{P}(r_j(n) = 0)$. □

Since, by Lemma 2.1, $r_i(n) = 0$ if, and only if, i is a divisor of n, it follows from Corollary 2.7 that:

Theorem 2.8. *If i and j are co-prime and $i \neq j$, then whether, or not, i divides any given natural number n is independent of whether, or not, j divides n.* □

Proof: (i) By Corollary 2.6, we have that:

$$\mathbb{P}((r_i(n) = 0) \cap (r_j(n) = 0)) = \frac{1}{i.j}$$

$$\mathbb{P}(r_i(n) = 0).\mathbb{P}(r_j(n) = 0) = (\tfrac{1}{i})(\tfrac{1}{j}).$$

[26] In other words, the compound probability of determining u and v correctly from the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die, is the product of the probability of determining u correctly from the roll of an i-sided cylindrical die, and the probability of determining v correctly from the roll of a j-sided cylindrical die.

(ii) Further, if i and j are co-prime, and $n+r_{i.j}(n) \equiv 0 \ (mod\ i.j)$, then the $i.j$ integers $r_j(n).i+r_i(n).j$ are all incongruent and form a complete system of residues. It follows that $n = a.i$—whence i divides n—and also $n = b.j$—whence j divides n—if, and only if $r_i(n) = r_j(n) = r_{i.j}(n) = 0$.

The lemma follows. □

We thus conclude that:

Corollary 2.9. *The prime divisors of any integer n are mutually independent.* □

2.C.a. Integer Factorising cannot be polynomial-time

We digress briefly from our investigation of prime counting functions to note that Theorem 2.8 immediately yields the computational complexity consequence[27] that no deterministic algorithm[28] can compute a factor of any randomly given integer n in polynomial time[29].

We note the standard definition[30]:

Definition 3. *A deterministic algorithm computes a number-theoretical function $f(n)$ in polynomial-time if there exists k such that, for all inputs n, the algorithm computes $f(n)$ in $\leq (log_e\ n)^k + k$ steps.*

It then follows from Corollary 2.9 that:

Corollary 2.10. *Any deterministic algorithm that always computes a prime factor of n cannot be polynomial-time.*

Proof: Any computational process that successfully identifies a prime divisor of n must necessarily appeal to at least one logical operation for identifying such a factor.

Since n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, and n may be the square of a prime, it follows from Corollary 2.9 that we necessarily require at least one logical operation for each prime $p \leq \sqrt{n}$ in order to logically determine whether p is a prime divisor of n.

Since the number of such primes is of the order $O(\sqrt{n}/log_e\ n)$, the number of computations required by any deterministic algorithm that always computes a prime factor of n cannot be polynomial-time—i.e. of order $O((log_e\ n)^c)$ for any c—in the length of the input n. The corollary follows. □

3. Density of integers not divisible by primes $Q = \{q_1, q_2, \ldots, q_k\}$

Reverting back to our consideration of prime distribution, we conclude from Lemma 2.2 and Corollary 2.9 that:

Lemma 3.1. *The asymptotic density of the set of all integers that are not divisible by any of a given set of primes $Q = \{q_1, q_2, \ldots, q_k\}$ is:*

$$\prod_{q \in Q}(1 - 1/q).$$ □

It follows that:

[27] cf. [Cook].

[28] A deterministic algorithm computes a mathematical function which has a unique value for any input in its domain, and the algorithm is a process that produces this particular value as output.

[29] cf. [Cook], p.1; also [Br00], p.1, fn.1.

[30] cf. [Cook], p.1; also [Br00], p.1, fn.1: “For a polynomial-time algorithm the expected running time should be a polynomial in the length of the input, i.e. $O((logN)^c)$ for some constant c”.

Lemma 3.2. *The expected number of integers in any interval (a,b) that are not divisible by any of a given set of primes* $Q = \{q_1, q_2, \ldots, q_k\}$ *is:*

$$(b-a)\prod_{q \in Q}(1-1/q). \qquad \square$$

3.A. The function $\pi_H(n)$

In particular, the expected number $\pi_H(n)$ of integers $\leq n$ that are not divisible by any of the first k primes $p_1, p_2, \ldots, p_k$ is:

Corollary 3.3. $\pi_H(n) = n.\prod_{i=1}^{k}(1 - \frac{1}{p_i})$.

It follows that:

Corollary 3.4. *The expected number of primes* $\leq p^2_{\pi(\sqrt{n})+1}$ *is:*

$$\pi_H(p^2_{\pi(\sqrt{n})+1}) = p^2_{\pi(\sqrt{n})+1}\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$$

with cumulative standard deviation:

$$p_{\pi(\sqrt{n})+1}\sqrt{\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})(1 - \prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i}))}. \qquad \square$$

We conclude that $\pi_H(n)$ is the non-heuristic approximation of the number of primes $\leq n$[31]:

Lemma 3.5. $\pi(n) \approx \pi_H(n) = n.\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$.

3.B. The function $\pi_L(n)$

It also follows immediately from Theorem 3.2 that:

Corollary 3.6. *The expected number of primes in the interval* $(p^2_{\pi(\sqrt{n})},\ p^2_{\pi(\sqrt{n})+1})$ *is:*

$$(p^2_{\pi(\sqrt{n})+1} - p^2_{\pi(\sqrt{n})})\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$$

with standard binomial deviation:

$$\sqrt{(p^2_{\pi(\sqrt{n})+1} - p^2_{\pi(\sqrt{n})})\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})(1 - \prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i}))}. \qquad \square$$

It further follows from Lemma 3.2 and Corollary 3.6 that:

Corollary 3.7. *The number* $\pi(p^2_{\pi(\sqrt{n})+1})$ *of primes less than* $p^2_{\pi(\sqrt{n})+1}$ *is cumulatively approximated by:*

$$\pi_L(p^2_{\pi(\sqrt{n})+1}) = \sum_{j=1}^{\pi(\sqrt{n})}\{(p^2_{j+1} - p^2_j)\prod_{i=1}^{j}(1 - \frac{1}{p_i})\}$$

with cumulative standard deviation:

$$\sum_{j=1}^{\pi(\sqrt{n})}\sqrt{(p^2_{j+1} - p^2_j)\prod_{i=1}^{j}(1 - \frac{1}{p_i})(1 - \prod_{i=1}^{j}(1 - \frac{1}{p_i}))}. \qquad \square$$

[31]Figs.12 in §7. compares the values of $\pi(n)$ and $\pi_H(n)$ for $4 \leq n \leq 1500$.

We conclude that $\pi_L(n)$ is the cumulative non-heuristic approximation of the number of primes $\leq n$[32]:

Lemma 3.8. $\pi(n) \approx \pi_L(n) = \sum_{j=1}^{n} \prod_{i=1}^{\pi(\sqrt{j})} (1 - \frac{1}{p_i})$.

It immediately follows from Lemma 3.5 and Lemma 3.8 that:

Corollary 3.9. *$\pi_L(n) > \pi_H(n)$ for all $n \geq 9$.*

3.C. The interval $(p_n^2,\ p_{n+1}^2)$

It follows immediately from the definition of $\pi(x)$ as the number of primes less than or equal to x that:

Lemma 3.10. $\prod_{i=1}^{\pi(\sqrt{x})} (1 - \frac{1}{p_i}) = \prod_{i=1}^{\pi(\sqrt{x+1})} (1 - \frac{1}{p_i})$ *for* $p_n^2 \leq x < p_{n+1}^2$. □

We can thus generalise the number-theoretic function of Lemma 3.8 as the real-valued function:

Definition 4. $\pi_L(x) = \pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^{n} (1 - \frac{1}{p_i})$ *for* $p_n^2 \leq x < p_{n+1}^2$. □

We note that the graph of $\pi_L(x)$ in the interval $(p_n^2,\ p_{n+1}^2)$ for $n \geq 1$ is now a straight line with gradient $\prod_{i=1}^{n} (1 - \frac{1}{p_i})$, as illustrated in §1.G., Fig.5 where we defined $\pi_L(x)$ equivalently by:

$$\pi_L(x) = (x - p_n^2) \prod_{i=1}^{n} (1 - \frac{1}{p_i}) + \sum_{j=1}^{n-1} (p_{j+1}^2 - p_j^2) \prod_{i=1}^{j} (1 - \frac{1}{p_i}) + 2$$

3.D. The functions $\pi_L(x)/\frac{x}{log_e x}$ and $\pi_H(x)/\frac{x}{log_e x}$

We consider next the function $\pi_L(x)/\frac{x}{log_e x}$ in the interval $(p_n^2,\ p_{n+1}^2)$:

$$\pi_L(x)/\frac{x}{log_e x} = (\pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^{n} (1 - \frac{1}{p_i}))/\frac{x}{log_e x}$$

This now yields the derivative $(\pi_L(x).\frac{log_e x}{x})'$ in the interval $(p_n^2,\ p_{n+1}^2)$ as:

$$\pi_L(x).(\frac{log_e x}{x})' + (\pi_L(x))'.\frac{log_e x}{x}$$

$$(\pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^{n} (1 - \frac{1}{p_i})).(\frac{log_e x}{x})' + (\pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^{n} (1 - \frac{1}{p_i}))'.\frac{log_e x}{x}$$

$$(\pi_L(p_n^2) + (x - p_n^2) \prod_{i=1}^{n} (1 - \frac{1}{p_i})).(\frac{1}{x^2} - \frac{log_e x}{x^2}) + (\prod_{i=1}^{n} (1 - \frac{1}{p_i})).\frac{log_e x}{x}$$

Since $p_n^2 \leq x < p_{n+1}^2$, by Mertens'[33] and Chebyshev's Theorems we can express the above as:

$$\sim (\pi_L(p_n^2) + \frac{e^{-\gamma}(x - p_n^2)}{log_e n}).(\frac{1}{x^2} - \frac{log_e x}{x^2}) + \frac{e^{-\gamma}.log_e x}{x.log_e n}$$

$$\sim (\frac{\pi_L(p_n^2)}{x} + \frac{e^{-\gamma}}{log_e n}(1 - \frac{p_n^2}{x})).\frac{(1 - log_e x)}{x} + \frac{e^{-\gamma}.log_e x}{x.log_e n}$$

$$\sim (\frac{\pi_L(p_n^2)}{p_n^2}.\frac{p_n^2}{x} + \frac{e^{-\gamma}}{log_e n}(1 - \frac{p_n^2}{x})).\frac{(1 - 2.log_e p_n)}{p_n^2} + \frac{2.e^{-\gamma}.log_e p_n}{p_n^2.log_e n}$$

Since each term $\rightarrow 0$ as $n \rightarrow \infty$, we conclude that the function $\pi_L(x)/\frac{x}{log_e x}$ does not oscillate but tends to a limit as $x \rightarrow \infty$ since:

Lemma 3.11. $(\pi_L(x)/\frac{x}{log_e x})' \in o(1)$. □

[32] Fig.12 in §7., and Fig.15 in §7.A., comparatively analyse the values of $\pi(n)$ and $\pi_L(n)$ for $4 \leq n \leq 1500$.
[33] [HW60], Theorem 429, p.351.

We further conclude that:

Corollary 3.12. $\pi_L(n) = \sum_{j=1}^{n} \prod_{i=1}^{\pi(\sqrt{j})}(1 - \frac{1}{p_i}) \sim a.\frac{n}{log_e n}$ *for some constant* a. $\square$

We note that $a > 2.e^{-\gamma}$[34], since $\prod_{i=1}^{\pi(\sqrt{j})}(1 - \frac{1}{p_i}) \geq \prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$ for all $1 \leq j \leq n$, and it follows from Definition 3.3 that:

Corollary 3.13. $\pi_H(n) = n.\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i}) \sim 2.e^{-\gamma}.\frac{n}{log_e n}$[35]. $\square$

4. Primes in an arithmetic progression

We consider now Dirichlet's Theorem, which is the assertion that if a and d are co-prime and $1 \leq a < d$, then the arithmetic progression $a + m.d$, where $m \geq 1$, contains an infinitude of (Dirichlet) primes.

We first note that, by Lemma 2.6:

Lemma 4.1. *If* p_i *and* p_j *are two primes where* $i \neq j$ *then, for any* $n \geq 2$, $\alpha, \beta \geq 1$, *we have:*

$$\mathbb{P}((r_{p_i^\alpha}(n) = u) \cap (r_{p_j^\beta}(n) = v)) = \mathbb{P}(r_{p_i^\alpha}(n) = u).\mathbb{P}(r_{p_j^\beta}(n) = v)$$

where $p_i^\alpha > u \geq 0$ *and* $p_j^\beta > v \geq 0$. $\square$

Now, the $p_i^\alpha.p_j^\beta$ numbers $d.p_i^\alpha + c.p_j^\beta$, where $p_i^\alpha > c \geq 0$ and $p_j^\beta > d \geq 0$, are all incongruent and form a complete system of residues[36] mod $(p_i^\alpha.p_j^\beta)$. It follows that $n = a.p_i^\alpha$—whence p_i^α divides n—and also $n = b.p_i^\beta$—whence p_i^β divides n—if, and only if $r_{p_i^\alpha}(n) = r_{p_i^\beta}(n) = 0$.

If $u = 0$ and $v = 0$ in Lemma 4.1, so that both p_i and p_j are prime divisors of n, we immediately conclude that:

$$\mathbb{P}((r_{p_i^\alpha}(n) = 0) \cap (r_{p_j^\beta}(n) = 0)) = \frac{1}{p_i^\alpha.p_j^\beta}$$

$$\mathbb{P}(r_{p_i^\alpha}(n) = 0).\mathbb{P}(r_{p_j^\beta}(n) = 0) = (\frac{1}{p_i^\alpha})(\frac{1}{p_j^\beta}).$$

Corollary 4.2. $\mathbb{P}((r_{p_i^\alpha}(n) = 0) \cap (r_{p_j^\beta}(n) = 0)) = \mathbb{P}(r_{p_i^\alpha}(n) = 0).\mathbb{P}(r_{p_j{}^\beta}(n) = 0)$. $\square$

It also immediately follows that Corollary 2.9 can be extended to prime powers in general[37]:

Theorem 4.3. *For any two primes* $p \neq q$ *and natural numbers* $n, \alpha, \beta \geq 1$, *whether or not* p^α *divides* n *is independent of whether or not* q^β *divides* n. $\square$

[34] Where $2.e^{-\lambda} \approx 1.12292\ldots$; [Gr95], p.13.
[35] By Mertens' Theorem; since $log_e\pi(\sqrt{n}) \sim (log_e\sqrt{n} - log_e log_e\sqrt{n})$ by the Prime Number Theorem.
[36] [HW60], p.52, Theorem 59.
[37] *Hint*: The following arguments may be easier to follow if we visualise the residues $r_{p_i^\alpha}(n)$ and $r_{p_i^\beta}(n)$ as they would occur in §6., Fig.7 and Fig.8.

4.A. The asymptotic density of Dirichlet integers

We note next that:

Lemma 4.4. *For any co-prime natural numbers* $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_k}$ *where:*

$$q_1 < q_2 < \ldots < q_k \text{ are primes and } \alpha_1, \alpha_2 \ldots \alpha_k \geq 1 \text{ are natural numbers;}$$

the natural number n *is of the form* $a + m.d$ *for some natural number* $m \geq 1$ *if, and only if:*

$$a + r_{q_i^{\alpha_i}}(n) \equiv 0 \ (mod\ q_i^{\alpha_i}) \text{ for all } 1 \leq i \leq k$$

where $0 \leq r_i(n) < i$ *is defined for all* $i > 1$ *by:*

$$n + r_i(n) \equiv 0 \ (mod\ i)\ .$$

Proof: First, if n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_k}$, then:

$$\begin{array}{rrcll} & n & \equiv & a\ (mod\ d) & \\ and: & n + r_{q_i^{\alpha_i}}(n) & \equiv & 0\ (mod\ q_i^{\alpha_i}) & for\ all\ 1 \leq i \leq k \\ whence: & a + r_{q_i^{\alpha_i}}(n) & \equiv & 0\ (mod\ q_i^{\alpha_i}) & for\ all\ 1 \leq i \leq k \end{array}$$

Second:

$$\begin{array}{rrcll} If: & a + r_{q_i^{\alpha_i}}(n) & \equiv & 0\ (mod\ q_i^{\alpha_i}) & for\ all\ 1 \leq i \leq k \\ and: & n + r_{q_i^{\alpha_i}}(n) & \equiv & 0\ (mod\ q_i^{\alpha_i}) & for\ all\ 1 \leq i \leq k \\ then: & n - a & \equiv & 0\ (mod\ q_i^{\alpha_i}) & for\ all\ 1 \leq i \leq k \\ whence: & n & \equiv & a\ (mod\ d) & \end{array}$$

The Lemma follows. □

By Lemma 2.2, it follows that:

Corollary 4.5. *The probability that* $a + r_{q_i^{\alpha_i}}(n) \equiv 0\ (mod\ q_i^{\alpha_i})$ *for any* $1 \leq i \leq k$ *is* $\frac{1}{q_i^{\alpha_i}}$. □

By Lemma 4.1 and Theorem 4.3, it further follows that:

Corollary 4.6. *The joint probability that* $a + r_{q_i^{\alpha_i}}(n) \equiv 0\ (mod\ q_i^{\alpha_i})$ *for all* $1 \leq i \leq k$ *is* $\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}}$. □

We conclude by Lemma 4.4 that:

Corollary 4.7. *The asymptotic density of Dirichlet integers, defined as numbers of the form* $a + m.d$ *for some natural number* $m \geq 1$ *which are not divisible by any given set of primes* $\mathbb{R} = \{r_1, r_2, \ldots, r_l\}$, *where* $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_l}$ *is:*

$$\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} . \prod_{r \in \mathbb{R}\ \&\ r \neq q_i} (1 - \frac{1}{r}).$$

Proof: Since a, d are co-prime, we have by Lemma 4.4 that if n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, we have that:

$$\begin{array}{rlcll} & n & \equiv & a \ (mod\ q_i) & for\ all\ 1 \leq i \leq k \\ whilst: & n + r_i(n) & \equiv & 0 \ (mod\ i) & for\ all\ 1 \leq i \\ whence: & a + r_{q_i}(n) & \equiv & 0 \ (mod\ q_i) & for\ all\ 1 \leq i \leq k \\ & r_{q_i}(n) & \neq & 0 & for\ all\ 1 \leq i \leq k \\ and: & q_i & \not| & n & for\ all\ 1 \leq i \leq k \end{array}$$

Hence, if n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2} \ldots q_k^{\alpha_k}$ and $(a, d) = 1$, the probability that $q_i \not| n$ for all $1 \leq i \leq k$ is 1.

By Lemma 3.1, Theorem 3.2 and Theorem 4.3, the asymptotic density of Dirichlet numbers of the form $a + m.d$ which are not divisible by any given set of primes $\mathbb{R} = \{r_1, r_2, \ldots, r_l\}$ is thus:

$$\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} . \prod_{r \in \mathbb{R}\ \&\ r \neq q_i} (1 - \frac{1}{r})$$

The Corollary follows. □

Corollary 4.8. *The expected number of Dirichlet integers in any interval (a, b) is:*

$$(b - a) \prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} . \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} . \prod_{r \in \mathbb{R}} (1 - \frac{1}{r}).$$ □

4.B. An elementary non-heuristic proof of Dirichlet's Theorem

Since n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, it follows that the number $\pi_{(a,d)}(n)$ of Dirichlet primes, of the form $a + m.d$ for some natural number $m \geq 1$ and $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, that are less than or equal to any $n \geq q_k^2$ is cumulatively approximated by the non-heuristic Dirichlet prime counting function:

Definition 5. $\pi_D(n) = \sum_{l=1}^{n} (\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} . \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} . \prod_{j=1}^{\pi(\sqrt{l})} (1 - \frac{1}{p_j}).$

We conclude that:

Lemma 4.9. $\pi_{(a,d)}(n) \approx \pi_D(n) \rightarrow \infty$ *as* $n \rightarrow \infty$.

Proof: If a, d are co-prime and $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, we have for any $n \geq q_k^2$:

$$\begin{aligned} \pi_D(n) &= \sum_{l=1}^{n} (\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} . \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} . \prod_{j=1}^{\pi(\sqrt{l})} (1 - \frac{1}{p_j})) \\ &= \prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} . \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} . \sum_{l=1}^{n} \prod_{j=1}^{\pi(\sqrt{l})} (1 - \frac{1}{p_j}) \\ &\geq \prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} . \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} . n . \prod_{j=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_j}) \end{aligned}$$

The lemma follows since, by Mertens' Theorem, $\prod_{p \leq x} (1 - \frac{1}{p}) \sim \frac{e^{-\lambda}}{log_e x}$, we have that:

$$n.\prod_{j=1}^{\pi(\sqrt{n})}(1-\frac{1}{p_j}) \sim \frac{2e^{-\gamma}n}{log_e(n)} \rightarrow \infty \text{ as } n \rightarrow \infty. \quad \square$$

Since $p_{n+1}^2 - p_n^2 \rightarrow \infty$ as $n \rightarrow \infty$, we conclude that:

Theorem 4.10. *There are an infinity of primes in any arithmetic progression* $a+m.d$ *where* $(a,d) = 1$[38]. $\square$

5. An elementary non-heuristic proof that there are infinitely many twin-primes

We define $\pi_2(n)$ as the number of integers $p \leq n$ such that both p and $p+2$ are prime.

In order to estimate $\pi_2(n)$, we first define:

Definition 6. *An integer* n *is a* $\mathbb{TW}(k)$ *integer if, and only if,* $r_{p_i}(n) \neq 0$ *and* $r_{p_i}(n) \neq 2$ *for all* $1 \leq i \leq k$*, where* $0 \leq r_i(n) < i$ *is defined for all* $i > 1$ *by:*

$$n + r_i(n) \equiv 0 \ (mod \ i) \ .$$

We note that:

Lemma 5.1. *If* n *is a* $\mathbb{TW}(k)$ *integer, then both* n *and* $n+2$ *are not divisible by any of the first* k *primes* $\{p_1, p_2, \ldots, p_k\}$.

Proof: The lemma follows immediately from Definition 6, Definition 1 and Lemma 2.1. $\square$

Since each residue $r_i(n)$ cycles over the i values $(i-1, i-2, \ldots, 0)$, these values are all incongruent and form a complete system of residues *mod* i.

It thus follows from Definition 6 that the asymptotic density of $\mathbb{TW}(k)$ integers over the set of natural numbers is:

Lemma 5.2. $\mathbb{D}(\mathbb{TW}(k)) = \prod_{i=2}^{k}(1-\frac{2}{p_i})$. $\square$

We also have that:

Lemma 5.3. *If* $p_k^2 \leq n \leq p_{k+1}^2$ *is a* $\mathbb{TW}(k)$ *integer, then* n *is a prime and either* $n+2$ *is also a prime, or* $n+2 = p_{k+1}^2$.

Proof: By Definition 6 and Definition 1:

$$\begin{array}{rcl} r_{p_i}(n) & \neq & 2 \ for \ all \ 1 \leq i \leq k \\ n+2 & \neq & \lambda.p_i \ for \ all \ 1 \leq i \leq k, \ \lambda \geq 1 \end{array}$$

Hence n is prime; and either $n+2$ is divisible by p_{k+1}, in which case $n+2 = p_{k+1}^2$, or it is a prime. $\square$

If we define $\pi_{\mathbb{TW}(k)}(n)$ as the number of $\mathbb{TW}(k)$ integers $\leq n$, by Lemma 5.2 the expected number of $\mathbb{TW}(k)$ integers in any interval (a,b) is given—with a binomial standard deviation—by:

Lemma 5.4. $\pi_{\mathbb{TW}(k)}(b) - \pi_{\mathbb{TW}(k)}(a) \approx (b-a)\prod_{i=2}^{k}(1-\frac{2}{p_i})$. $\square$

[38] Compare [HW60], p.13, Theorem 15*.

Since n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, it follows from Lemma 5.3 that $\pi_{\mathbb{TW}(k)}(p_{k+1}^2) - \pi_{\mathbb{TW}(k)}(p_k^2)$ is at most one less than the number of twin-primes in the interval $(p_{k+1}^2 - p_k^2)$.

Lemma 5.5. $\pi_{\mathbb{TW}(k)}(p_{k+1}^2) - \pi_{\mathbb{TW}(k)}(p_k^2) + 1 \geq \pi_2(p_{k+1}^2) - \pi_2(p_{k)}^2) \geq \pi_{\mathbb{TW}(k)}(p_{k+1}^2) - \pi_{\mathbb{TW}(k)}(p_k^2)$

Now, by Lemma 5.4 the expected number of $\mathbb{TW}(k)$ integers in the interval $(p_{k+1}^2 - p_k^2)$ is given by:

Lemma 5.6. $\pi_{\mathbb{TW}(k)}(p_{k+1}^2) - \pi_{\mathbb{TW}(k)}(p_k^2) \approx (p_{k+1}^2 - p_k^2)\prod_{i=2}^{k}(1 - \frac{2}{p_i})$. □

We conclude that the number $\pi_2(p_{k+1}^2)$ of twin primes $\leq p_{k+1}^2$ is given by the cumulative non-heuristic approximation:

Lemma 5.7. $\sum_{j=1}^{k}(\pi_2(p_{j+1}^2) - \pi_2(p_j^2)) = \pi_2(p_{k+1}^2) \approx \sum_{j=1}^{k}(p_{j+1}^2 - p_j^2)\prod_{i=2}^{j}(1 - \frac{2}{p_i})$. □

We further conclude that:

Theorem 5.8. $\pi_2(n) \rightarrow \infty$ *as* $n \rightarrow \infty$.

Proof: We have that, for $k \geq 2$:

$$\begin{aligned}
\sum_{j=1}^{k}(p_{j+1}^2 - p_j^2)\prod_{i=2}^{j}(1 - \tfrac{2}{p_i}) &= \sum_{j=9}^{p_{k+1}^2}\prod_{i=2}^{\pi(\sqrt{j})-1}(1 - \tfrac{2}{p_i}) \\
&\geq (p_{k+1}^2 - 9).\prod_{i=2}^{k}(1 - \tfrac{2}{p_i}) \\
&\geq (p_{k+1}^2 - 9).\prod_{i=2}^{k}(1 - \tfrac{1}{p_i})(1 - \tfrac{1}{(p_i-1)}) \\
&\geq (p_{k+1}^2 - 9).\prod_{i=2}^{k}(1 - \tfrac{1}{p_i})(1 - \tfrac{1}{p_{i-1}}) \\
&\geq (p_{k+1}^2 - 9).\prod_{i=2}^{k}(1 - \tfrac{1}{p_{i-1}})^2 \\
&\geq (p_{k+1}^2 - 9).\prod_{i=1}^{k}(1 - \tfrac{1}{p_i})^2
\end{aligned}$$

Now, by Mertens' Theorem, we have that:

$$\begin{aligned}
(p_{k+1}^2 - 9).\prod_{i=1}^{k}(1 - \tfrac{1}{p_i})^2 &\sim (p_{k+1}^2 - 9).(\tfrac{e^{-\gamma}}{log_e k})^2 \\
&\rightarrow \infty \ as \ n \rightarrow \infty
\end{aligned}$$

The theorem follows by Lemma 5.7. □

5.A. The Generalised Prime Counting Function: $\sum_{j=1}^{n}\prod_{i=a}^{\pi(\sqrt{j})}(1 - \frac{b}{p_i})$

We note that the argument of Theorem 5.8 in §5. is a special case of the limiting behaviour of the Generalised Prime Counting Function $\sum_{j=1}^{n}\prod_{i=a}^{\pi(\sqrt{j})}(1 - \frac{b}{p_i})$, which estimates the number of integers $\leq n$ such that there are b values that cannot occur amongst the residues $r_{p_i}(n)$ for $a \leq i \leq \pi(\sqrt{j})$[39]:

[39]Thus $b = 1$ yields an estimate for the number of primes $\leq n$, and $b = 2$ an estimate for the number of TW primes (Definition 6) $\leq n$.

Theorem 5.9. $\sum_{j=1}^{n}\prod_{i=a}^{\pi(\sqrt{j})}(1-\frac{b}{p_i}) \rightarrow \infty$ *as* $n \rightarrow \infty$ *if* $p_a > b \geq 1$.

Proof: For $p_a > b \geq 1$, we have that:

$$\begin{aligned}
\sum_{j=1}^{n}\prod_{i=a}^{\pi(\sqrt{j})}(1-\tfrac{b}{p_i}) &\geq \sum_{j=p_a^2}^{n}\prod_{i=a}^{\pi(\sqrt{j})}(1-\tfrac{b}{p_i}) \\
&\geq \sum_{j=p_a^2}^{n}\prod_{i=a}^{\pi(\sqrt{n})}(1-\tfrac{b}{p_i}) \\
&\geq (n-p_a^2).\prod_{i=a}^{\pi(\sqrt{n})}(1-\tfrac{b}{p_i}) \\
&\geq (n-p_a^2).\prod_{i=a}^{n}(1-\tfrac{b}{p_i})
\end{aligned}$$

The theorem follows if:

$$log_e(n-p_a^2) + \sum_{i=a}^{n} log_e(1-\tfrac{b}{p_i}) \rightarrow \infty$$

(i) We note first the standard result for $|x| < 1$ that:

$$log_e(1-x) = -\sum_{m=1}^{\infty}\tfrac{x^m}{m}$$

For any $p_i > b \geq 1$, we thus have:

$$log_e(1-\tfrac{b}{p_i}) = -\sum_{m=1}^{\infty}\tfrac{(b/p_i)^m}{m} = -\tfrac{b}{p_i} - \sum_{m=2}^{\infty}\tfrac{(b/p_i)^m}{m}$$

Hence:

$$\sum_{i=a}^{n} log_e(1-\tfrac{b}{p_i}) = -\sum_{i=a}^{n}(\tfrac{b}{p_i}) - \sum_{i=a}^{n}(\sum_{m=2}^{\infty}\tfrac{(b/p_i)^m}{m})$$

(ii) We note next that, for all $i \geq a$:

$$c < (1-\tfrac{b}{p_a}) \rightarrow c < (1-\tfrac{b}{p_i})$$

It follows for any such c that:

$$\sum_{m=2}^{\infty}\tfrac{(b/p_i)^m}{m} \leq \sum_{m=2}^{\infty}(\tfrac{b}{p_i})^m = \tfrac{(b/p_i)^2}{1-b/p_i} \leq \tfrac{b^2}{c.p_i^2}$$

Since:

$$\sum_{i=1}^{\infty}\tfrac{1}{p_i^2} = O(1)$$

it further follows that:

$$\sum_{i=a}^{n}(\sum_{m=2}^{\infty}\tfrac{(b/p_i)^m}{m}) \leq \sum_{i=a}^{n}(\tfrac{b^2}{c.p_i^2}) = O(1)$$

(iii) From the standard result[40]:

$$\sum_{p \leq x} \frac{1}{p} = log_e log_e x + O(1) + o(1)$$

it then follows that:

$$\begin{aligned} \sum_{i=a}^{n} log_e(1 - \frac{b}{p_i}) &\geq -\sum_{i=a}^{n}(\frac{b}{p_i}) - O(1) \\ &\geq -b.(log_e log_e n + O(1) + o(1)) - O(1) \end{aligned}$$

The theorem follows since:

$$log_e(n - p_a^2) - b.(log_e log_e n + O(1) + o(1)) - O(1) \rightarrow \infty$$

and so:

$$log_e(n - p_a^2) + \sum_{i=a}^{n} log_e(1 - \frac{b}{p_i}) \rightarrow \infty$$ □

6. Appendix I: The residue function $r_i(n)$

We graphically illustrate how the residues $r_i(n)$ occur naturally as values of:

A: The natural-number based residue sequences R_i;

B: The natural-number based residue sequences $E(n)$;

and as the output of:

C: The natural-number based algorithm $E_{\mathbb{N}}$;

D: The prime-number based algorithm $E_{\mathbb{P}}$;

E: The prime-number based algorithm $E_{\mathbb{Q}}$.

A: The natural-number based sequences $R_i(n)$

Density: For instance, the residues $r_i(n)$ can be defined for all $n \geq 1$ as the values of the sequences $R_i(n)$, defined for all $i \geq 1$, as illustrated below in Fig.7[41], where:

- For any $i \geq 2$, each sequence $R_i(n)$ cycles through the values $(i-1,\ i-2,\ \ldots,\ 0)$ with period i;
- For any $i \geq 2$ the asymptotic density—over the set of natural numbers—of the set $\{n\}$ of integers that are divisible by i is $\frac{1}{i}$; and the asymptotic density of integers that are not divisible by i is $\frac{i-1}{i}$.

Sequence:	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	$\ldots R_n$
$n=1$	0	1	2	3	4	5	6	7	8	9	10	...n-1
$n=2$	0	0	1	2	3	4	5	6	7	8	9	...n-2
$n=3$	0	1	0	1	2	3	4	5	6	7	8	...n-3
$n=4$	0	0	2	0	1	2	3	4	5	6	7	...n-4
$n=5$	0	1	1	3	0	1	2	3	4	5	6	...n-5
$n=6$	0	0	0	2	4	0	1	2	3	4	5	...n-6
$n=7$	0	1	2	1	3	5	0	1	2	3	4	...n-7
$n=8$	0	0	1	0	2	4	6	0	1	2	3	...n-8
$n=9$	0	1	0	3	1	3	5	7	0	1	2	...n-9
$n=10$	0	0	2	2	0	2	4	6	8	0	1	...n-10
$n=11$	0	1	1	1	4	1	3	5	7	9	0	...n-11
n	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	$\ldots 0$

Fig.7: The natural-number based residue sequences $R_i(n)$

40 [HW60], p.351, Theorem 427.

41 For r_i read $r_i(n)$; for R_i read $R_i(n)$.

B: The natural-number based sequences $E(n)$

Primality: The residues $r_i(n)$ can also be viewed alternatively as values of the associated sequences, $E(n) = \{r_i(n) : i \geq 1\}$, defined for all $n \geq 1$, as illustrated below in Fig.8, where:

- The sequences $E(n)$ highlighted in red correspond to a prime[42] p (since $r_i(p) \neq 0$ for $1 < i < p$) in the usual, linearly displayed, Eratosthenes sieve:

  $E(\not{1})$, $E(2)$, $E(3)$, $E(\not{4})$, $E(5)$, $E(\not{6})$, $E(7)$, $E(\not{8})$, $E(\not{9})$, $E(\not{10})$, $E(11)$, ...

- The sequences highlighted in cyan identify a crossed out composite n (since $r_i(n) = 0$ for some $i < i < n$) in the usual, linearly displayed, Eratosthenes sieve.

- The 'boundary' residues $r_1(n) = 0$ and $r_n(n) = 0$ are identified in cyan.

Sequence:	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	$\dots R_n$
$E(1)$:	0	1	2	3	4	5	6	7	8	9	10	...n-1
$E(2)$:	0	0	1	2	3	4	5	6	7	8	9	...n-2
$E(3)$:	0	1	0	1	2	3	4	5	6	7	8	...n-3
$E(4)$:	0	0	2	0	1	2	3	4	5	6	7	...n-4
$E(5)$:	0	1	1	3	0	1	2	3	4	5	6	...n-5
$E(6)$:	0	0	0	2	4	0	1	2	3	4	5	...n-6
$E(7)$:	0	1	2	1	3	5	0	1	2	3	4	...n-7
$E(8)$:	0	0	1	0	2	4	6	0	1	2	3	...n-8
$E(9)$:	0	1	0	3	1	3	5	7	0	1	2	...n-9
$E(10)$:	0	0	2	2	0	2	4	6	8	0	1	...n-10
$E(11)$:	0	1	1	1	4	1	3	5	7	9	0	...n-11
...												
$E(n)$:	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	...0
...												

Fig.8: The natural-number based residue sequences $E(n)$

C: The output of a natural-number based algorithm $E_{\mathbb{N}}$

We give below in Fig.9 the output for $1 \leq n \leq 11$ of a natural-number based algorithm $E_{\mathbb{N}}$ that computes the values $r_i(n)$ of the sequence $E_{\mathbb{N}}(n)$ for only $1 \leq i \leq n$ for any given n.

Divisors:	1	2	3	4	5	6	7	8	9	10	11	...n ...
$E_{\mathbb{N}}(1)$:	0											
$E_{\mathbb{N}}(2)$:	0	0										
$E_{\mathbb{N}}(3)$:	0	1	0									
$E_{\mathbb{N}}(4)$:	0	0	2	0								
$E_{\mathbb{N}}(5)$:	0	1	1	3	0							
$E_{\mathbb{N}}(6)$:	0	0	0	2	4	0						
$E_{\mathbb{N}}(7)$:	0	1	2	1	3	5	0					
$E_{\mathbb{N}}(8)$:	0	0	1	0	2	4	6	0				
$E_{\mathbb{N}}(9)$:	0	1	0	3	1	3	5	7	0			
$E_{\mathbb{N}}(10)$:	0	0	2	2	0	2	4	6	8	0		
$E_{\mathbb{N}}(11)$:	0	1	1	1	4	1	3	5	7	9	0	
...												
$E_{\mathbb{N}}(n)$:	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	...0
...												

Fig.9: The output of the natural-number based algorithm $E_{\mathbb{N}}$

D: The output of the prime-number based algorithm $E_{\mathbb{P}}$

Fig.10 gives the output for $2 \leq n \leq 31$ of a prime-number based algorithm $E_{\mathbb{Q}}$ that computes the values $q_i(n) = r_{p_i}(n)$ of the sequence $E_{\mathbb{P}}(n)$ for only each prime $2 \leq p_i \leq n$ for any given n.

Prime: Divisor:	p_1 2	p_2 3	p_3 5	p_4 7	p_5 11	p_6 13	p_7 17	p_8 19	p_9 23	p_{10} 29	p_{11} 31	$\dots p_n \dots$ $\dots p_n \dots$
$E_{\mathbb{P}}(2)$:	0											
$E_{\mathbb{P}}(3)$:	1	0										
$E_{\mathbb{P}}(4)$:	0	2										
$E_{\mathbb{P}}(5)$:	1	1	0									
$E_{\mathbb{P}}(6)$:	0	0	4									
$E_{\mathbb{P}}(7)$:	1	2	3	0								
$E_{\mathbb{P}}(8)$:	0	1	2	6								
$E_{\mathbb{P}}(9)$:	1	0	1	5								
$E_{\mathbb{P}}(10)$:	0	2	0	4								
$E_{\mathbb{P}}(11)$:	1	1	4	3	0							
$E_{\mathbb{P}}(12)$:	0	0	3	2	10							
$E_{\mathbb{P}}(13)$:	1	2	2	1	9	0						

[42] Conventionally defined as integers that are not divisible by any smaller integer other than 1.

$E_{\mathbb{P}}(14)$:	0	1	1	0	8	12						
$E_{\mathbb{P}}(15)$:	1	0	0	6	7	11						
$E_{\mathbb{P}}(16)$:	0	2	4	5	6	10						
$E_{\mathbb{P}}(17)$:	1	1	3	4	5	9	0					
$E_{\mathbb{P}}(18)$:	0	0	2	3	4	8	16					
$E_{\mathbb{P}}(19)$:	1	2	1	2	3	7	15	0				
$E_{\mathbb{P}}(20)$:	0	1	0	1	2	6	14	18				
$E_{\mathbb{P}}(21)$:	1	0	4	0	1	5	13	17				
$E_{\mathbb{P}}(22)$:	0	2	3	6	0	4	12	16				
$E_{\mathbb{P}}(23)$:	1	1	2	5	10	3	11	15	0			
$E_{\mathbb{P}}(24)$:	0	0	1	4	9	2	10	14	22			
$E_{\mathbb{P}}(25)$:	1	2	0	3	8	1	9	13	21			
$E_{\mathbb{P}}(26)$:	0	1	4	2	7	0	8	12	20			
$E_{\mathbb{P}}(27)$:	1	0	3	1	6	12	7	11	19			
$E_{\mathbb{P}}(28)$:	0	2	2	0	5	11	6	10	18			
$E_{\mathbb{P}}(29)$:	1	1	1	6	4	10	5	9	17	0		
$E_{\mathbb{P}}(30)$:	0	0	0	5	3	9	4	8	16	28		
$E_{\mathbb{P}}(31)$:	1	2	4	4	2	8	3	7	15	27	0	
...												
$E_{\mathbb{P}}(n)$:	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	$\ldots 0$
...												

Fig.10: The output of the prime-number based algorithm $E_{\mathbb{P}}$

E: The output of the prime-number based algorithms $E_{\mathbb{P}}$ and $E_{\mathbb{Q}}$

We give below in Fig.11 the output for $2 \leq n \leq 121$ of the two prime-number based algorithms $E_{\mathbb{P}}$ (whose output $\{q_i(n) = r_{p_i}(n) : 1 \leq i \leq \pi(n)\}$ is shown only partially, partly in cyan) and $E_{\mathbb{Q}}$ (whose output $q_i(n) = \{r_{p_i}(n) : 1 \leq i \leq \pi(\sqrt{n})\}$ is highlighted in black and red, the latter indicating the generation of a prime sequence[43].

Prime:	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	$\ldots p_n \ldots$
Divisor:	2	3	5	7	11	13	17	19	23	29	31	$\ldots p_n \ldots$
Function:	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	$\ldots$
$E_{\mathbb{Q}}(2)$:	0	(Prime by definition)										
$E_{\mathbb{Q}}(3)$:	1	0										
$E_{\mathbb{Q}}(4)$:	0	2										
$E_{\mathbb{Q}}(5)$:	1	1	0									
$E_{\mathbb{Q}}(6)$:	0	0	4									
$E_{\mathbb{Q}}(7)$:	1	2	3	0								
$E_{\mathbb{Q}}(8)$:	0	1	2	6								
$E_{\mathbb{Q}}(9)$:	1	0	1	5								
$E_{\mathbb{Q}}(10)$:	0	2	0	4								
$E_{\mathbb{Q}}(11)$:	1	1	4	3	0							
$E_{\mathbb{Q}}(12)$:	0	0	3	2	10							
$E_{\mathbb{Q}}(13)$:	1	2	2	1	9	0						
$E_{\mathbb{Q}}(14)$:	0	1	1	0	8	12						
$E_{\mathbb{Q}}(15)$:	1	0	0	6	7	11						
$E_{\mathbb{Q}}(16)$:	0	2	4	5	6	10						
$E_{\mathbb{Q}}(17)$:	1	1	3	4	5	9	0					
$E_{\mathbb{Q}}(18)$:	0	0	2	3	4	8	16					
$E_{\mathbb{Q}}(19)$:	1	2	1	2	3	7	15	0				
$E_{\mathbb{Q}}(20)$:	0	1	0	1	2	6	14	18				
$E_{\mathbb{Q}}(21)$:	1	0	4	0	1	5	13	17				
$E_{\mathbb{Q}}(22)$:	0	2	3	6	0	4	12	16				
$E_{\mathbb{Q}}(23)$:	1	1	2	5	10	3	11	15	0			
$E_{\mathbb{Q}}(24)$:	0	0	1	4	9	2	10	14	22			
$E_{\mathbb{Q}}(25)$:	1	2	0	3	8	1	9	13	21			
$E_{\mathbb{Q}}(26)$:	0	1	4	2	7	0	8	12	20			
$E_{\mathbb{Q}}(27)$:	1	0	3	1	6	12	7	11	19			
$E_{\mathbb{Q}}(28)$:	0	2	2	0	5	11	6	10	18			
$E_{\mathbb{Q}}(29)$:	1	1	1	6	4	10	5	9	17	0		
$E_{\mathbb{Q}}(30)$:	0	0	0	5	3	9	4	8	16	28		
$E_{\mathbb{Q}}(31)$:	1	2	4	4	2	8	3	7	15	27	0	
$E_{\mathbb{Q}}(32)$:	0	1	3	3	1	7	2	6	14	26	30	
$E_{\mathbb{Q}}(33)$:	1	0	2	2	0	6	1	5	13	25	29	
$E_{\mathbb{Q}}(34)$:	0	2	1	1	10	5	0	4	12	24	28	
$E_{\mathbb{Q}}(35)$:	1	1	0	0	9	4	16	3	11	23	27	
$E_{\mathbb{Q}}(36)$:	0	0	4	6	8	3	15	2	10	22	26	
$E_{\mathbb{Q}}(37)$:	1	2	3	5	7	2	14	1	9	21	25	
$E_{\mathbb{Q}}(38)$:	0	1	2	4	6	1	13	0	8	20	24	
$E_{\mathbb{Q}}(39)$:	1	0	1	3	5	0	12	18	7	19	23	
$E_{\mathbb{Q}}(40)$:	0	2	0	2	4	12	11	17	6	18	22	
$E_{\mathbb{Q}}(41)$:	1	1	4	1	3	11	10	16	5	17	21	
$E_{\mathbb{Q}}(42)$:	0	0	3	0	2	10	9	15	4	16	20	
$E_{\mathbb{Q}}(43)$:	1	2	2	6	1	9	8	14	3	15	19	
$E_{\mathbb{Q}}(44)$:	0	1	1	5	0	8	7	13	2	14	18	
$E_{\mathbb{Q}}(45)$:	1	0	0	4	10	7	6	12	1	13	17	
$E_{\mathbb{Q}}(46)$:	0	2	4	3	9	6	5	11	0	12	16	
$E_{\mathbb{Q}}(47)$:	1	1	3	2	8	5	4	10	22	11	15	
$E_{\mathbb{Q}}(48)$:	0	0	2	1	7	4	3	9	21	10	14	
$E_{\mathbb{Q}}(49)$:	1	2	1	0	6	3	2	8	20	9	13	
$E_{\mathbb{Q}}(50)$:	0	1	0	6	5	2	1	7	19	8	12	
$E_{\mathbb{Q}}(51)$:	1	0	4	5	4	1	0	6	18	7	11	
$E_{\mathbb{Q}}(52)$:	0	2	3	4	3	0	16	5	17	6	10	
$E_{\mathbb{Q}}(53)$:	1	1	2	3	2	12	15	4	16	5	9	

[43]For informal reference and perspective, formal definitions of both the prime-number based algorithms $E_{\mathbb{P}}$ and $E_{\mathbb{Q}}$ are given in this work in progress *Factorising all $m \leq n$ is of order $\Theta(\sum_{i=2}^{n} \pi(\sqrt{i}))$.*

$E_{\mathbb{Q}}(54)$:	0	0	1	2	1	11	14	3	15	4	8	
$E_{\mathbb{Q}}(55)$:	1	2	0	1	0	10	13	2	14	3	7	
$E_{\mathbb{Q}}(56)$:	0	1	4	0	10	9	12	1	13	2	6	
$E_{\mathbb{Q}}(57)$:	1	0	3	6	9	8	11	0	12	1	5	
$E_{\mathbb{Q}}(58)$:	0	2	2	5	8	7	10	18	11	0	4	
$E_{\mathbb{Q}}(59)$:	1	1	1	4	7	6	9	17	10	28	3	
$E_{\mathbb{Q}}(60)$:	0	0	0	3	6	5	8	16	9	27	2	
$E_{\mathbb{Q}}(61)$:	1	2	4	2	5	4	7	15	8	26	1	
$E_{\mathbb{Q}}(62)$:	0	1	3	1	4	3	6	14	7	25	0	
$E_{\mathbb{Q}}(63)$:	1	0	2	0	3	2	5	13	6	24	30	
$E_{\mathbb{Q}}(64)$:	0	2	1	6	2	1	4	12	5	23	29	
$E_{\mathbb{Q}}(65)$:	1	1	0	5	1	0	3	11	4	22	28	
$E_{\mathbb{Q}}(66)$:	0	0	4	4	0	12	2	10	3	21	27	
$E_{\mathbb{Q}}(67)$:	1	2	3	3	10	11	1	9	2	20	26	
$E_{\mathbb{Q}}(68)$:	0	1	2	2	9	10	0	8	1	19	25	
$E_{\mathbb{Q}}(69)$:	1	0	1	1	8	9	16	7	0	18	24	
$E_{\mathbb{Q}}(70)$:	0	2	0	0	7	8	15	6	22	17	23	
$E_{\mathbb{Q}}(71)$:	1	1	4	6	6	7	14	5	21	16	22	
$E_{\mathbb{Q}}(72)$:	0	0	3	5	5	6	13	4	20	15	21	
$E_{\mathbb{Q}}(73)$:	1	2	2	4	4	5	12	3	19	14	20	
$E_{\mathbb{Q}}(74)$:	0	1	1	3	3	4	11	2	18	13	19	
$E_{\mathbb{Q}}(75)$:	1	0	0	2	2	3	10	1	17	12	18	
$E_{\mathbb{Q}}(76)$:	0	2	4	1	1	2	9	0	16	11	17	
$E_{\mathbb{Q}}(77)$:	1	1	3	0	0	1	8	18	15	10	16	
$E_{\mathbb{Q}}(78)$:	0	0	2	6	10	0	7	17	14	9	15	
$E_{\mathbb{Q}}(79)$:	1	2	1	5	9	12	6	16	13	8	14	
$E_{\mathbb{Q}}(80)$:	0	1	0	4	8	11	5	15	12	7	13	
$E_{\mathbb{Q}}(81)$:	1	0	4	3	7	10	4	14	11	6	12	
$E_{\mathbb{Q}}(82)$:	0	2	3	2	6	9	3	13	10	5	11	
$E_{\mathbb{Q}}(83)$:	1	1	2	1	5	8	2	12	9	4	10	
$E_{\mathbb{Q}}(84)$:	0	0	1	0	4	7	1	11	8	3	9	
$E_{\mathbb{Q}}(85)$:	1	2	0	6	3	6	0	10	7	2	8	
$E_{\mathbb{Q}}(86)$:	0	1	4	5	2	5	16	9	6	1	7	
$E_{\mathbb{Q}}(87)$:	1	0	3	4	1	4	15	8	5	0	6	
$E_{\mathbb{Q}}(88)$:	0	2	2	3	0	3	14	7	4	28	5	
$E_{\mathbb{Q}}(89)$:	1	1	1	2	10	2	13	6	3	27	4	
$E_{\mathbb{Q}}(90)$:	0	0	0	1	9	1	12	5	2	26	3	
$E_{\mathbb{Q}}(91)$:	1	2	4	0	8	0	11	4	1	25	2	
$E_{\mathbb{Q}}(92)$:	0	1	3	6	7	12	10	3	0	24	1	
$E_{\mathbb{Q}}(93)$:	1	0	2	5	6	11	9	2	22	23	0	
$E_{\mathbb{Q}}(94)$:	0	2	1	4	5	10	8	1	21	22	30	
$E_{\mathbb{Q}}(95)$:	1	1	0	3	4	9	7	0	20	21	29	
$E_{\mathbb{Q}}(96)$:	0	0	4	2	3	8	6	18	19	20	28	
$E_{\mathbb{Q}}(97)$:	1	2	3	1	2	7	5	17	18	19	27	
$E_{\mathbb{Q}}(98)$:	0	1	2	0	1	6	4	16	17	18	26	
$E_{\mathbb{Q}}(99)$:	1	0	1	6	0	5	3	15	16	17	25	
$E_{\mathbb{Q}}(100)$:	0	2	0	5	10	4	2	14	15	16	24	
$E_{\mathbb{Q}}(101)$:	1	1	4	4	9	3	1	13	14	15	23	
$E_{\mathbb{Q}}(102)$:	0	0	3	3	8	2	0	12	13	14	22	
$E_{\mathbb{Q}}(103)$:	1	2	2	2	7	1	16	11	12	13	21	
$E_{\mathbb{Q}}(104)$:	0	1	1	1	6	0	15	10	11	12	20	
$E_{\mathbb{Q}}(105)$:	1	0	0	0	5	12	14	9	10	11	19	
$E_{\mathbb{Q}}(106)$:	0	2	4	6	4	11	13	8	9	10	18	
$E_{\mathbb{Q}}(107)$:	1	1	3	5	3	10	12	7	8	9	17	
$E_{\mathbb{Q}}(108)$:	0	0	2	4	2	9	11	6	7	8	16	
$E_{\mathbb{Q}}(109)$:	1	2	1	3	1	8	10	5	6	7	15	
$E_{\mathbb{Q}}(110)$:	0	1	0	2	0	7	9	4	5	6	14	
$E_{\mathbb{Q}}(111)$:	1	0	4	1	10	6	8	3	4	5	13	
$E_{\mathbb{Q}}(112)$:	0	2	3	0	9	5	7	2	3	4	12	
$E_{\mathbb{Q}}(113)$:	1	1	2	6	8	4	6	1	2	3	11	
$E_{\mathbb{Q}}(114)$:	0	0	1	5	7	3	5	0	1	2	10	
$E_{\mathbb{Q}}(115)$:	1	2	0	4	6	2	4	18	0	1	9	
$E_{\mathbb{Q}}(116)$:	0	1	4	3	5	1	3	17	22	0	8	
$E_{\mathbb{Q}}(117)$:	1	0	3	2	4	0	2	16	21	28	7	
$E_{\mathbb{Q}}(118)$:	0	2	2	1	3	12	1	15	20	27	6	
$E_{\mathbb{Q}}(119)$:	1	1	1	0	2	11	0	14	19	26	5	
$E_{\mathbb{Q}}(120)$:	0	0	0	6	1	10	16	13	18	25	4	
$E_{\mathbb{Q}}(121)$:	1	2	4	5	0	9	15	12	17	24	3	
...												
$E_{\mathbb{Q}}(n)$:	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	q_{11}	...
...												
Prime:	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	$\ldots p_n \ldots$
Divisor:	2	3	5	7	11	13	17	19	23	29	31	$\ldots p_n \ldots$

Fig.11: The output of the prime-number based algorithms $E_{\mathbb{P}}$ and $E_{\mathbb{Q}}$

7. Appendix II: Comparative values of actual and non-heuristically estimated number of primes ≤ 3000

Based on the manual and spreadsheet calculations detailed in Fig.14 below, the following two graph compare the non-heuristically estimated values of:

(i) $\pi_H(n) = \sum_{j=1}^{n} \prod_{i=1}^{\sqrt{n}} (1 - \frac{1}{p_i}) = n. \prod_{i=1}^{\sqrt{n}} (1 - \frac{1}{p_i})$ (green); and

(ii) $\pi_L(n) = \sum_{j=1}^{n} \prod_{i=1}^{\sqrt{j}} (1 - \frac{1}{p_i})$ (red);

vs the actual values of $\pi(n)$ (blue) for $4 \leq n \leq 1500$ and $4 \leq n \leq 3000$.

Fig.12: Comparative non-heuristically estimated distributions of the primes ≤ 1500

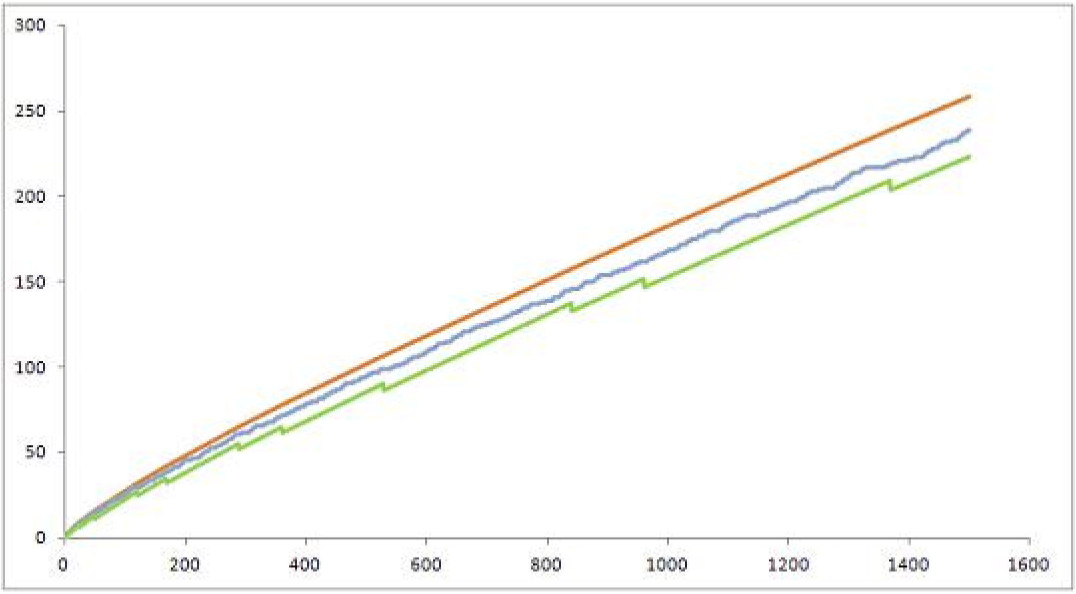

Fig.12: The above graph compares the non-heuristically estimated values of $\pi_H(n) = \sum_{j=1}^{n}\prod_{i=1}^{\sqrt{n}}(1-\frac{1}{p_i}) = n.\prod_{i=1}^{\sqrt{n}}(1-\frac{1}{p_i})$ (green) and $\pi_L(n) = \sum_{j=1}^{n}\prod_{i=1}^{\sqrt{j}}(1-\frac{1}{p_i})$ (red) vs the actual values of $\pi(n)$ (blue) for $4 \leq n \leq 1500$.

Fig.13: Comparative non-heuristically estimated distributions of the primes ≤ 3000

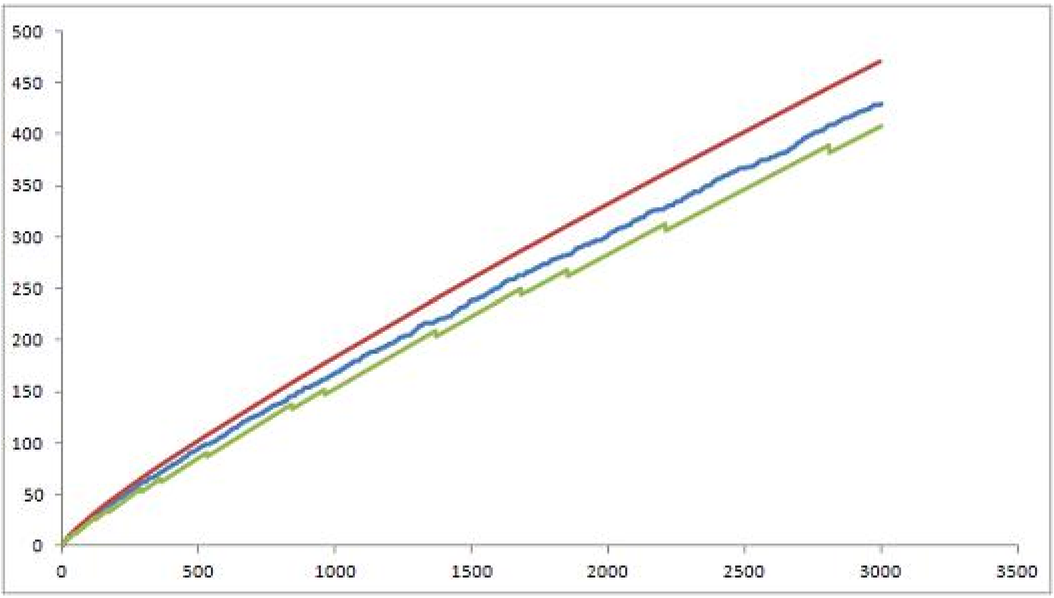

Fig.13: The above graph compares the non-heuristically estimated values of $\pi_H(n) = \sum_{j=1}^{n}\prod_{i=1}^{\sqrt{n}}(1-\frac{1}{p_i}) = n.\prod_{i=1}^{\sqrt{n}}(1-\frac{1}{p_i})$ (green) and $\pi_L(n) = \sum_{j=1}^{n}\prod_{i=1}^{\sqrt{j}}(1-\frac{1}{p_i})$ (red) vs the actual values of $\pi(n)$ (blue) for $4 \leq n \leq 3000$.

Since $\pi(n) \sim \frac{n}{log_e(n)}$ by the Prime Number Theorem, whilst $\pi_H(n) \sim 2e^{-\lambda}\frac{n}{log_e n}$ where $2.e^{-\gamma} \approx 1.12292\ldots$, and $\pi_L(n) > \pi_H(n)$ for all $n \geq 9$ by Corollary 3.9, this raises the interesting queries:

(a) Which is the least n such that $\pi_H(n) > \pi(n)$?

(b) Which is the largest n such that $\pi(n) > \pi_H(n)$?

Fig.14: The following table gives comparative values for $\pi(n)$ as approximated non-heuristically by $\pi_L(n) = \sum_{j=1}^{n} \prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$, the actual values $\pi(n)$ of the primes less than or equal to n, and the values for $\pi(n)$ as estimated non-heuristically by $\pi_H(n) = n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$ of $\pi(n)$, for $4 \leq n \leq 1500$.

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
1	2	0	1/2	1		0.0000	0	0.000
2	3	0	1/3	1	1/2	0.5000	1	1.0000
3	5	0	4/15	1	1/2	1.0000	2	1.5000
4	7	1	8/35	2	1/3	1.3333	2	1.3333
5	11	1	16/77	2	1/3	1.6667	3	1.6667
6	13	1	89/464	2	1/3	2.0000	3	2.0000
7	17	1	165/914	2	1/3	2.3333	4	2.3333
8	19	1	157/918	2	1/3	2.6667	4	2.6667
9	23	2	62/379	3	4/15	2.9333	4	2.4000
10	29	2	157/994	3	4/15	3.2000	4	2.6667
11	31	2	142/929	3	4/15	3.4667	5	2.9333
12	37	2	29/195	3	4/15	3.7333	5	3.2000
13	41	2	139/958	3	4/15	4.0000	6	3.4667
14	43	2	89/628	3	4/15	4.2667	6	3.7333
15	47	2	62/447	3	4/15	4.5333	6	4.0000
16	53	2	112/823	4	4/15	4.8000	6	4.2667
17	59	2	40/299	4	4/15	5.0667	7	4.5333
18	61	2	5/38	4	4/15	5.3333	7	4.8000
19	67	2	7/54	4	4/15	5.6000	8	5.0667
20	71	2	40/313	4	4/15	5.8667	8	5.3333
21	73	2	15/119	4	4/15	6.1333	8	5.6000
22	79	2	85/683	4	4/15	6.4000	8	5.8667
23	83	2	15/122	4	4/15	6.6667	9	6.1333
24	89	2	31/255	4	4/15	6.9333	9	6.4000
25	97	3	106/881	5	8/35	7.1619	9	5.7143
26	101	3	109/915	5	8/35	7.3905	9	5.9429
27	103	3	86/729	5	8/35	7.6190	9	6.1714
28	107	3	97/830	5	8/35	7.8476	9	6.4000
29	109	3	11/95	5	8/35	8.0762	10	6.6286
30	113	3	7/61	5	8/35	8.3048	10	6.8751
31	127	3	101/887	5	8/35	8.5333	11	7.0857
32	131	3	20/177	5	8/35	8.7619	11	7.3143
33	137	3	47/419	5	8/35	8.9905	11	7.5429
34	139	3	49/440	5	8/35	9.2190	11	7.7714
35	149	3	25/226	5	8/35	9.4476	11	8.0000
36	151	3	10/91	6	8/35	9.6762	11	8.2286
37	157	3	63/577	6	8/35	9.9048	12	8.4571
38	163	3	79/728	6	8/35	10.1333	12	8.6857
39	167	3	48/445	6	8/35	10.3619	12	8.9143
40	173	3	77/718	6	8/35	10.5905	12	9.1429
41	179	3	61/572	6	8/35	10.8190	13	9.3714
42	181	3	7/66	6	8/35	11.0476	13	9.6000
43	191	3	94/891	6	8/35	11.2762	14	9.8286
44	193	3	89/848	6	8/35	11.5048	14	10.0571
45	197	3	26/249	6	8/35	11.7333	14	10.2857
46	199	3	8/77	6	8/35	11.9619	14	10.5143
47	211	3	76/735	6	8/35	12.1905	15	10.7429
48	223	3	7/68	6	8/35	12.4190	15	10.9714
49	227	4	33/322	7	16/77	12.6268	15	10.1818
50	229	4	5/49	7	16/77	12.8346	15	10.3896
51	233	4	19/187	7	16/77	13.0424	15	10.5974
52	239	4	43/425	7	16/77	13.2502	15	10.8052
53	241	4	40/397	7	16/77	13.4580	16	11.0130
54	251	4	57/568	7	16/77	13.6658	16	11.2208
55	257	4	1/10	7	16/77	13.8736	16	11.4286
56	263	4	24/241	7	16/77	14.0814	16	11.6364
57	269	4	63/635	7	16/77	14.2892	16	11.8442
58	271	4	60/607	7	16/77	14.4970	16	12.0519
59	277	4	13/132	7	16/77	14.7048	17	12.2597
60	281	4	58/591	7	16/77	14.9126	17	12.4675
61	283	4	31/317	7	16/77	15.1203	18	12.6753
62	293	4	23/236	7	16/77	15.3281	18	12.8831
63	307	4	17/175	7	16/77	15.5359	18	13.0909
64	311	4	58/599	8	16/77	15.7437	18	13.2987
65	313	4	61/632	8	16/77	15.9515	18	13.5065
66	317	4	61/634	8	16/77	16.1593	18	13.7143
67	331	4	40/417	8	16/77	16.3671	19	13.9221
68	337	4	68/711	8	16/77	16.5749	19	14.1299
69	347	4	72/755	8	16/77	16.7827	19	14.3377
70	349	4	31/326	8	16/77	16.9905	19	14.5455
71	353	4	11/116	8	16/77	17.1983	20	14.7532
72	359	4	33/349	8	16/77	17.4061	20	14.9610
73	367	4	43/456	8	16/77	17.6139	21	15.1688
74	373	4	79/840	8	16/77	17.8216	21	15.3766
75	379	4	59/629	8	16/77	18.0294	21	15.5844
76	383	4	45/481	8	16/77	18.2372	21	15.7922
77	389	4	60/643	8	16/77	18.4450	21	16.0000
78	397	4	39/419	8	16/77	18.6528	21	16.2078
79	401	4	61/657	8	16/77	18.8606	22	16.4156

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
80	409	4	64/691	8	16/77	19.0684	22	16.6234
81	419	4	79/855	9	16/77	19.2762	22	16.8312
82	421	4	33/358	9	16/77	19.4840	22	17.0390
83	431	4	8/87	9	16/77	19.6918	23	17.2468
84	433	4	89/970	9	16/77	19.8996	23	17.4545
85	439	4	13/142	9	16/77	20.1074	23	17.6623
86	443	4	39/427	9	16/77	20.3152	23	17.8701
87	449	4	37/406	9	16/77	20.5229	23	18.0779
88	457	4	1/11	9	16/77	20.7307	23	18.2857
89	461	4	48/529	9	16/77	20.9385	24	18.4935
90	463	4	67/740	9	16/77	21.1463	24	18.7013
91	467	4	73/808	9	16/77	21.3541	24	18.9091
92	479	4	11/122	9	16/77	21.5619	24	19.1169
93	487	4	35/389	9	16/77	21.7697	24	19.3247
94	491	4	51/568	9	16/77	21.9775	24	19.5325
95	499	4	69/770	9	16/77	22.1853	24	19.7403
96	503	4	11/123	9	16/77	22.3931	24	19.9481
97	509	4	54/605	9	16/77	22.6009	25	20.1558
98	521	4	71/797	9	16/77	22.8087	25	20.3636
99	523	4	77/866	9	16/77	23.0165	25	20.5714
100	541	4	71/800	10	16/77	23.2242	25	20.7792
101	547	4	59/666	10	16/77	23.4320	26	20.9870
102	557	4	81/916	10	16/77	23.6398	26	21.1948
103	563	4	73/827	10	16/77	23.8476	27	21.4026
104	569	4	43/488	10	16/77	24.0554	27	21.6104
105	571	4	19/216	10	16/77	24.2632	27	21.8182
106	577	4	85/968	10	16/77	24.4710	27	22.0260
107	587	4	27/308	10	16/77	24.6788	28	22.2338
108	593	4	7/80	10	16/77	24.8866	28	22.4416
109	599	4	65/744	10	16/77	25.0944	29	22.6494
110	601	4	43/493	10	16/77	25.3022	29	22.8571
111	607	4	31/356	10	16/77	25.5100	29	23.0649
112	613	4	2/23	10	16/77	25.7177	29	23.2727
113	617	4	23/265	10	16/77	25.9255	30	23.4805
114	619	4	50/577	10	16/77	26.1333	30	23.6883
115	631	4	77/890	10	16/77	26.3411	30	23.8961
116	641	4	85/984	10	16/77	26.5489	30	24.1039
117	643	4	37/429	10	16/77	26.7567	30	24.3117
118	647	4	80/929	10	16/77	26.9645	30	24.5195
119	653	4	46/535	10	16/77	27.1723	30	24.7273
120	659	4	54/629	10	16/77	27.3801	30	24.9351
121	661	5	3/35	11	89/464	27.5719	30	23.2088
122	673	5	41/479	11	89/464	27.7637	30	23.4006
123	677	5	10/117	11	89/464	27.9555	30	23.5924
124	683	5	46/539	11	89/464	28.1473	30	23.7842
125	691	5	64/751	11	89/464	28.3391	30	23.9760
126	701	5	4/47	11	89/464	28.5309	30	24.1678
127	709	5	56/659	11	89/464	28.7227	31	24.3596
128	719	5	51/601	11	89/464	28.9146	31	24.5514
129	727	5	5/59	11	89/464	29.1064	31	24.7433
130	733	5	60/709	11	89/464	29.2982	31	24.9351
131	739	5	6/71	11	89/464	29.4900	32	25.1269
132	743	5	33/391	11	89/464	29.6818	32	25.3187
133	751	5	59/700	11	89/464	29.8736	32	25.5105
134	757	5	25/297	11	89/464	30.0654	32	25.7023
135	761	5	67/797	11	89/464	30.2572	32	25.8941
136	769	5	45/536	11	89/464	30.4490	32	26.0859
137	773	5	68/811	11	89/464	30.6408	33	26.2777
138	787	5	17/203	11	89/464	30.8326	33	26.4695
139	797	5	23/275	11	89/464	31.0244	34	26.6613
140	809	5	35/419	11	89/464	31.2163	34	26.8531
141	811	5	73/875	11	89/464	31.4081	34	27.0450
142	821	5	1/12	11	89/464	31.5999	34	27.2368
143	823	5	64/769	11	89/464	31.7917	34	27.4286
144	827	5	33/397	12	89/464	31.9835	34	27.6204
145	829	5	45/542	12	89/464	32.1753	34	27.8122
146	839	5	17/205	12	89/464	32.3671	34	28.0040
147	853	5	41/495	12	89/464	32.5589	34	28.1958
148	857	5	23/278	12	89/464	32.7507	34	28.3876
149	859	5	79/956	12	89/464	32.9425	35	28.5794
150	863	5	26/315	12	89/464	33.1343	35	28.7712
151	877	5	31/376	12	89/464	33.3261	36	28.9630
152	881	5	7/85	12	89/464	33.5179	36	29.1548
153	883	5	51/620	12	89/464	33.7098	36	29.3467
154	887	5	47/572	12	89/464	33.9016	36	29.5385
155	907	5	49/597	12	89/464	34.0934	36	29.7303
156	911	5	71/866	12	89/464	34.2852	36	29.9221
157	919	5	19/232	12	89/464	34.4770	37	30.1139
158	929	5	76/929	12	89/464	34.6688	37	30.3057
159	937	5	38/465	12	89/464	34.8606	37	30.4975
160	941	5	4/49	12	89/464	35.0524	37	30.6893
161	947	5	19/233	12	89/464	35.2442	37	30.8811
162	953	5	29/356	12	89/464	35.4360	37	31.0729
163	967	5	52/639	12	89/464	35.6278	38	31.2647
164	971	5	10/123	12	89/464	35.8196	38	31.4565
165	977	5	51/628	12	89/464	36.0115	38	31.6484
166	983	5	46/567	12	89/464	36.2033	38	31.8402
167	991	5	62/765	12	89/464	36.3951	39	32.0320
168	997	5	57/704	12	89/464	36.5869	39	32.2238
169	1009	6	11/136	13	165/914	36.7674	39	30.5088
170	1013	6	8/99	13	165/914	36.9479	39	30.6893
171	1019	6	49/607	13	165/914	37.1285	39	30.8698
172	1021	6	5/62	13	165/914	37.3090	39	31.0504

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
173	1031	6	17/211	13	165/914	37.4895	40	31.2309
174	1033	6	59/733	13	165/914	37.6700	40	31.4114
175	1039	6	39/485	13	165/914	37.8506	40	31.5919
176	1049	6	67/834	13	165/914	38.0311	40	31.7725
177	1051	6	37/461	13	165/914	38.2116	40	31.9530
178	1061	6	17/212	13	165/914	38.3921	40	32.1335
179	1063	6	29/362	13	165/914	38.5727	41	32.3140
180	1069	6	2/25	13	165/914	38.7532	41	32.4946
181	1087	6	79/988	13	165/914	38.9337	42	32.6751
182	1091	6	56/701	13	165/914	39.1142	42	32.8556
183	1093	6	17/213	13	165/914	39.2948	42	33.0361
184	1097	6	37/464	13	165/914	39.4753	42	33.2167
185	1103	6	29/364	13	165/914	39.6558	42	33.3972
186	1109	6	71/892	13	165/914	39.8363	42	33.5777
187	1117	6	47/591	13	165/914	40.0169	42	33.7582
188	1123	6	70/881	13	165/914	40.1974	42	33.9388
189	1129	6	62/781	13	165/914	40.3779	42	34.1193
190	1151	6	51/643	13	165/914	40.5584	42	34.2998
191	1153	6	21/265	13	165/914	40.7390	43	34.4803
192	1163	6	27/341	13	165/914	40.9195	43	34.6609
193	1171	6	25/316	13	165/914	41.1000	44	34.8414
194	1181	6	43/544	13	165/914	41.2805	44	35.0219
195	1187	6	68/861	13	165/914	41.4611	44	35.2024
196	1193	6	58/735	14	165/914	41.6416	44	35.3830
197	1201	6	41/520	14	165/914	41.8221	45	35.5635
198	1213	6	62/787	14	165/914	42.0026	45	35.7440
199	1217	6	27/343	14	165/914	42.1832	46	35.9245
200	1223	6	7/89	14	165/914	42.3637	46	36.1051
201	1229	6	69/878	14	165/914	42.5442	46	36.2856
202	1231	6	34/433	14	165/914	42.7247	46	36.4661
203	1237	6	55/701	14	165/914	42.9053	46	36.6466
204	1249	6	45/574	14	165/914	43.0858	46	36.8272
205	1259	6	64/817	14	165/914	43.2663	46	37.0077
206	1277	6	49/626	14	165/914	43.4468	46	37.1882
207	1279	6	14/179	14	165/914	43.6274	46	37.3687
208	1283	6	44/563	14	165/914	43.8079	46	37.5493
209	1289	6	36/461	14	165/914	43.9884	46	37.7298
210	1291	6	65/833	14	165/914	44.1689	46	37.9103
211	1297	6	63/808	14	165/914	44.3495	47	38.0909
212	1301	6	6/77	14	165/914	44.5300	47	38.2714
213	1303	6	71/912	14	165/914	44.7105	47	38.4519
214	1307	6	62/797	14	165/914	44.8910	47	38.6324
215	1319	6	37/476	14	165/914	45.0716	47	38.8130
216	1321	6	8/103	14	165/914	45.2521	47	38.9935
217	1327	6	69/889	14	165/914	45.4326	47	39.1740
218	1361	6	47/606	14	165/914	45.6131	47	39.3545
219	1367	6	31/400	14	165/914	45.7937	47	39.5351
220	1373	6	57/736	14	165/914	45.9742	47	39.7156
221	1381	6	64/827	14	165/914	46.1547	47	39.8961
222	1399	6	29/375	14	165/914	46.3352	47	40.0766
223	1409	6	67/867	14	165/914	46.5158	48	40.2572
224	1423	6	59/764	14	165/914	46.6963	48	40.4377
225	1427	6	24/311	15	165/914	46.8768	48	40.6182
226	1429	6	31/402	15	165/914	47.0574	48	40.7987
227	1433	6	43/558	15	165/914	47.2379	49	40.9793
228	1439	6	69/896	15	165/914	47.4184	49	41.1598
229	1447	6	1/13	15	165/914	47.5989	50	41.3403
230	1451	6	1/13	15	165/914	47.7795	50	41.5208
231	1453	6	1/13	15	165/914	47.9600	50	41.7014
232	1459	6	47/612	15	165/914	48.1405	50	41.8819
233	1471	6	33/430	15	165/914	48.3210	51	42.0624
234	1481	6	51/665	15	165/914	48.5016	51	42.2429
235	1483	6	21/274	15	165/914	48.6821	51	42.4235
236	1487	6	53/692	15	165/914	48.8626	51	42.6040
237	1489	6	61/797	15	165/914	49.0431	51	42.7845
238	1493	6	27/353	15	165/914	49.2237	51	42.9650
239	1499	6	12/157	15	165/914	49.4042	52	43.1456
240	1511	6	11/144	15	165/914	49.5847	52	43.3261
241	1523	6	10/131	15	165/914	49.7652	53	43.5066
242	1531	6	46/603	15	165/914	49.9458	53	43.6871
243	1543	6	17/223	15	165/914	50.1263	53	43.8677
244	1549	6	8/105	15	165/914	50.3068	53	44.0482
245	1553	6	67/880	15	165/914	50.4873	53	44.2287
246	1559	6	7/92	15	165/914	50.6679	53	44.4092
247	1567	6	53/697	15	165/914	50.8484	53	44.5898
248	1571	6	69/908	15	165/914	51.0289	53	44.7703
249	1579	6	6/79	15	165/914	51.2094	53	44.9508
250	1583	6	17/224	15	165/914	51.3900	53	45.1313
251	1597	6	38/501	15	165/914	51.5705	54	45.3119
252	1601	6	26/343	15	165/914	51.7510	54	45.4924
253	1607	6	5/66	15	165/914	51.9315	54	45.6729
254	1609	6	67/885	15	165/914	52.1121	54	45.8534
255	1613	6	23/304	15	165/914	52.2926	54	46.0340
256	1619	6	71/939	16	165/914	52.4731	54	46.2145
257	1621	6	30/397	16	165/914	52.6536	55	46.3950
258	1627	6	29/384	16	165/914	52.8342	55	46.5755
259	1637	6	4/53	16	165/914	53.0147	55	46.7561
260	1657	6	66/875	16	165/914	53.1952	55	46.9366
261	1663	6	64/849	16	165/914	53.3757	55	47.1171
262	1667	6	11/146	16	165/914	53.5563	55	47.2976
263	1669	6	71/943	16	165/914	53.7368	56	47.4782
264	1693	6	38/505	16	165/914	53.9173	56	47.6587
265	1697	6	37/492	16	165/914	54.0978	56	47.8392

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
266	1699	6	59/785	16	165/914	54.2784	56	48.0197
267	1709	6	16/213	16	165/914	54.4589	56	48.2003
268	1721	6	53/706	16	165/914	54.6394	56	48.3808
269	1723	6	67/893	16	165/914	54.8199	57	48.5613
270	1733	6	3/40	16	165/914	55.0005	57	48.7418
271	1741	6	32/427	16	165/914	55.1810	58	48.9224
272	1747	6	37/494	16	165/914	55.3615	58	49.1029
273	1753	6	39/521	16	165/914	55.5420	58	49.2834
274	1759	6	30/401	16	165/914	55.7226	58	49.4639
275	1777	6	8/107	16	165/914	55.9031	58	49.6445
276	1783	6	55/736	16	165/914	56.0836	58	49.8250
277	1787	6	18/241	16	165/914	56.2641	59	50.0055
278	1789	6	58/777	16	165/914	56.4447	59	50.1860
279	1801	6	47/630	16	165/914	56.6252	59	50.3666
280	1811	6	17/228	16	165/914	56.8057	59	50.5471
281	1823	6	74/993	16	165/914	56.9862	60	50.7276
282	1831	6	61/819	16	165/914	57.1668	60	50.9082
283	1847	6	30/403	16	165/914	57.3473	61	51.0887
284	1861	6	59/793	16	165/914	57.5278	61	51.2692
285	1867	6	67/901	16	165/914	57.7083	61	51.4497
286	1871	6	11/148	16	165/914	57.8889	61	51.6303
287	1873	6	13/175	16	165/914	58.0694	61	51.8108
288	1877	6	49/660	16	165/914	58.2499	61	51.9913
289	1879	7	21/283	17	157/918	58.4209	61	49.4259
290	1889	7	31/418	17	157/918	58.5920	61	49.5970
291	1901	7	55/742	17	157/918	58.7630	61	49.7680
292	1907	7	2/27	17	157/918	58.9340	61	49.9390
293	1913	7	2/27	17	157/918	59.1050	62	50.1100
294	1931	7	41/554	17	157/918	59.2761	62	50.2811
295	1933	7	27/365	17	157/918	59.4471	62	50.4521
296	1949	7	19/257	17	157/918	59.6181	62	50.6231
297	1951	7	15/203	17	157/918	59.7891	62	50.7941
298	1973	7	63/853	17	157/918	59.9602	62	50.9652
299	1979	7	11/149	17	157/918	60.1312	62	51.1362
300	1987	7	47/637	17	157/918	60.3022	62	51.3072
301	1993	7	25/339	17	157/918	60.4732	62	51.4782
302	1997	7	30/407	17	157/918	60.6443	62	51.6493
303	1999	7	61/828	17	157/918	60.8153	62	51.8203
304	2003	7	31/421	17	157/918	60.9863	62	51.9913
305	2011	7	63/856	17	157/918	61.1573	62	52.1623
306	2017	7	32/435	17	157/918	61.3284	62	52.3334
307	2027	7	5/68	17	157/918	61.4994	63	52.5044
308	2029	7	28/381	17	157/918	61.6704	63	52.6754
309	2039	7	57/776	17	157/918	61.8414	63	52.8464
310	2053	7	29/395	17	157/918	62.0125	63	53.0174
311	2063	7	51/695	17	157/918	62.1835	64	53.1885
312	2069	7	41/559	17	157/918	62.3545	64	53.3595
313	2081	7	64/873	17	157/918	62.5255	65	53.5305
314	2083	7	17/232	17	157/918	62.6965	65	53.7015
315	2087	7	26/355	17	157/918	62.8676	65	53.8726
316	2089	7	50/683	17	157/918	63.0386	65	54.0436
317	2099	7	3/41	17	157/918	63.2096	66	54.2146
318	2111	7	52/711	17	157/918	63.3806	66	54.3856
319	2113	7	25/342	17	157/918	63.5517	66	54.5567
320	2129	7	35/479	17	157/918	63.7227	66	54.7277
321	2131	7	13/178	17	157/918	63.8937	66	54.8987
322	2137	7	10/137	17	157/918	64.0647	66	55.0697
323	2141	7	17/233	17	157/918	64.2358	66	55.2408
324	2143	7	52/713	18	157/918	64.4068	66	55.4118
325	2153	7	39/535	18	157/918	64.5778	66	55.5828
326	2161	7	29/398	18	157/918	64.7488	66	55.7538
327	2179	7	26/357	18	157/918	64.9199	66	55.9249
328	2203	7	19/261	18	157/918	65.0909	66	56.0959
329	2207	7	35/481	18	157/918	65.2619	66	56.2669
330	2213	7	4/55	18	157/918	65.4329	66	56.4379
331	2221	7	45/619	18	157/918	65.6040	67	56.6090
332	2237	7	21/289	18	157/918	65.7750	67	56.7800
333	2239	7	56/771	18	157/918	65.9460	67	56.9510
334	2243	7	31/427	18	157/918	66.1170	67	57.1220
335	2251	7	50/689	18	157/918	66.2881	67	57.2930
336	2267	7	14/193	18	157/918	66.4591	67	57.4641
337	2269	7	53/731	18	157/918	66.6301	68	57.6351
338	2273	7	5/69	18	157/918	66.8011	68	57.8061
339	2281	7	46/635	18	157/918	66.9721	68	57.9771
340	2287	7	58/801	18	157/918	67.1432	68	58.1482
341	2293	7	49/677	18	157/918	67.3142	68	58.3192
342	2297	7	62/857	18	157/918	67.4852	68	58.4902
343	2309	7	35/484	18	157/918	67.6562	68	58.6612
344	2311	7	6/83	18	157/918	67.8273	68	58.8323
345	2333	7	25/346	18	157/918	67.9983	68	59.0033
346	2339	7	13/180	18	157/918	68.1693	68	59.1743
347	2341	7	27/374	18	157/918	68.3403	69	59.3453
348	2347	7	7/97	18	157/918	68.5114	69	59.5164
349	2351	7	22/305	18	157/918	68.6824	70	59.6874
350	2357	7	23/319	18	157/918	68.8534	70	59.8584
351	2371	7	8/111	18	157/918	69.0244	70	60.0294
352	2377	7	17/236	18	157/918	69.1955	70	60.2005
353	2381	7	71/986	18	157/918	69.3665	71	60.3715
354	2383	7	28/389	18	157/918	69.5375	71	60.5425
355	2389	7	10/139	18	157/918	69.7085	71	60.7135
356	2393	7	21/292	18	157/918	69.8796	71	60.8846
357	2399	7	56/779	18	157/918	70.0506	71	61.0556
358	2411	7	12/167	18	157/918	70.2216	71	61.2266

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
359	2417	7	13/181	18	157/918	70.3926	72	61.3976
360	2423	7	14/195	18	157/918	70.5637	72	61.5686
361	2437	8	15/209	19	62/379	70.7272	72	59.0553
362	2441	8	33/460	19	62/379	70.8908	72	59.2189
363	2447	8	18/251	19	62/379	71.0544	72	59.3825
364	2459	8	61/851	19	62/379	71.2180	72	59.5461
365	2467	8	23/321	19	62/379	71.3816	72	59.7097
366	2473	8	53/740	19	62/379	71.5452	72	59.8733
367	2477	8	31/433	19	62/379	71.7088	73	60.0369
368	2503	8	37/517	19	62/379	71.8724	73	60.2005
369	2521	8	47/657	19	62/379	72.0359	73	60.3640
370	2531	8	64/895	19	62/379	72.1995	73	60.5276
371	2539	8	1/14	19	62/379	72.3631	73	60.6912
372	2543	8	1/14	19	62/379	72.5267	73	60.8548
373	2549	8	1/14	19	62/379	72.6903	74	61.0184
374	2551	8	1/14	19	62/379	72.8539	74	61.1820
375	2557	8	1/14	19	62/379	73.0175	74	61.3456
376	2579	8	58/813	19	62/379	73.1811	74	61.5092
377	2591	8	44/617	19	62/379	73.3447	74	61.6727
378	2593	8	71/996	19	62/379	73.5082	74	61.8363
379	2609	8	30/421	19	62/379	73.6718	75	61.9999
380	2617	8	26/365	19	62/379	73.8354	75	62.1635
381	2621	8	68/955	19	62/379	73.9990	75	62.3271
382	2633	8	20/281	19	62/379	74.1626	75	62.4907
383	2647	8	18/253	19	62/379	74.3262	76	62.6543
384	2657	8	50/703	19	62/379	74.4898	76	62.8179
385	2659	8	61/858	19	62/379	74.6534	76	62.9815
386	2663	8	14/197	19	62/379	74.8169	76	63.1450
387	2671	8	13/183	19	62/379	74.9805	76	63.3086
388	2677	8	37/521	19	62/379	75.1441	76	63.4722
389	2683	8	23/324	19	62/379	75.3077	77	63.6358
390	2687	8	11/155	19	62/379	75.4713	77	63.7994
391	2689	8	31/437	19	62/379	75.6349	77	63.9630
392	2693	8	49/691	19	62/379	75.7985	77	64.1266
393	2699	8	28/395	19	62/379	75.9621	77	64.2902
394	2707	8	9/127	19	62/379	76.1257	77	64.4537
395	2711	8	17/240	19	62/379	76.2892	77	64.6173
396	2713	8	65/918	19	62/379	76.4528	77	65.7809
397	2719	8	39/551	19	62/379	76.6164	78	64.9445
398	2729	8	15/212	19	62/379	76.7800	78	65.1081
399	2731	8	36/509	19	62/379	76.9436	78	65.2717
400	2741	8	7/99	20	62/379	77.1072	78	65.4353
401	2749	8	47/665	20	62/379	77.2708	79	65.5989
402	2753	8	13/184	20	62/379	77.4344	79	65.7625
403	2767	8	44/623	20	62/379	77.5979	79	65.9260
404	2777	8	67/949	20	62/379	77.7615	79	66.0896
405	2789	8	59/836	20	62/379	77.9251	79	66.2532
406	2791	8	63/893	20	62/379	78.0887	79	66.4168
407	2797	8	39/553	20	62/379	78.2523	79	66.5804
408	2801	8	65/922	20	62/379	78.4159	79	66.7440
409	2803	8	58/823	20	62/379	78.5795	80	66.9076
410	2819	8	36/511	20	62/379	78.7431	80	67.0712
411	2833	8	5/71	20	62/379	78.9067	80	67.2347
412	2837	8	44/625	20	62/379	79.0702	80	67.3983
413	2843	8	62/881	20	62/379	79.2338	80	67.5619
414	2851	8	14/199	20	62/379	79.3974	80	67.7255
415	2857	8	41/583	20	62/379	79.5610	80	67.8891
416	2861	8	49/697	20	62/379	79.7246	80	68.0527
417	2879	8	61/868	20	62/379	79.8882	80	68.2163
418	2887	8	64/911	20	62/379	80.0518	80	68.3799
419	2897	8	46/655	20	62/379	80.2154	81	68.5435
420	2903	8	45/641	20	62/379	80.3789	81	68.7070
421	2909	8	4/57	20	62/379	80.5425	82	68.8706
422	2917	8	59/841	20	62/379	80.7061	82	69.0342
423	2927	8	27/385	20	62/379	80.8697	82	69.1978
424	2939	8	19/271	20	62/379	81.0333	82	69.3614
425	2953	8	67/956	20	62/379	81.1969	82	69.5250
426	2957	8	11/157	20	62/379	81.3605	82	69.6886
427	2963	8	18/257	20	62/379	81.5241	82	69.8522
428	2969	8	53/757	20	62/379	81.6876	82	70.0157
429	2971	8	66/943	20	62/379	81.8512	82	70.1793
430	2999	8	41/586	20	62/379	82.0148	82	70.3429
431	3001	8	37/529	20	62/379	82.1784	83	70.5065
432	3011	8	43/615	20	62/379	82.3420	83	70.6701
433	3019	8	13/186	20	62/379	82.5056	84	70.8337
434	3023	8	16/229	20	62/379	82.6692	84	70.9973
435	3037	8	19/272	20	62/379	82.8328	84	71.1609
436	3041	8	28/401	20	62/379	82.9964	84	71.3245
437	3049	8	43/616	20	62/379	83.1599	84	71.4880
438	3061	8	3/43	20	62/379	83.3235	84	71.6516
439	3067	8	3/43	20	62/379	83.4871	85	71.8152
440	3079	8	50/717	20	62/379	83.6507	85	71.9788
441	3083	8	29/416	21	62/379	83.8143	85	72.1424
442	3089	8	63/904	21	62/379	83.9779	85	72.3060
443	3109	8	65/933	21	62/379	84.1415	86	72.4696
444	3119	8	53/761	21	62/379	84.3051	86	72.6332
445	3121	8	11/158	21	62/379	84.4686	86	72.7967
446	3137	8	68/977	21	62/379	84.6322	86	72.9603
447	3163	8	43/618	21	62/379	84.7958	86	73.1239
448	3167	8	8/115	21	62/379	84.9594	86	73.2875
449	3169	8	21/302	21	62/379	85.1230	87	73.4511
450	3181	8	13/187	21	62/379	85.2866	87	73.6147
451	3187	8	41/590	21	62/379	85.4502	87	73.7783

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
452	3191	8	38/547	21	62/379	85.6138	87	73.9419
453	3203	8	5/72	21	62/379	85.7774	87	74.1055
454	3209	8	52/749	21	62/379	85.9409	87	74.2690
455	3217	8	49/706	21	62/379	86.1045	87	74.4326
456	3221	8	63/908	21	62/379	86.2681	87	74.5962
457	3229	8	12/173	21	62/379	86.4317	88	74.7598
458	3251	8	19/274	21	62/379	86.5953	88	74.9234
459	3253	8	54/779	21	62/379	86.7589	88	75.0870
460	3257	8	7/101	21	62/379	86.9225	88	75.2506
461	3259	8	23/332	21	62/379	87.0861	89	75.4142
462	3271	8	41/592	21	62/379	87.2496	89	75.5777
463	3299	8	9/130	21	62/379	87.4132	90	75.7413
464	3301	8	29/419	21	62/379	87.5768	90	75.9049
465	3307	8	42/607	21	62/379	87.7404	90	76.0685
466	3313	8	46/665	21	62/379	87.9040	90	76.2321
467	3319	8	13/188	21	62/379	88.0676	91	76.3957
468	3323	8	58/839	21	62/379	88.2312	91	76.5593
469	3329	8	66/955	21	62/379	88.3948	91	76.7229
470	3331	8	19/275	21	62/379	88.5584	91	76.8865
471	3343	8	23/333	21	62/379	88.7219	91	77.0500
472	3347	8	29/420	21	62/379	88.8855	91	77.2136
473	3359	8	39/565	21	62/379	89.0491	91	77.3772
474	3361	8	59/855	21	62/379	89.2127	91	77.5408
475	3371	8	2/29	21	62/379	89.3763	91	77.7044
476	3373	8	2/29	21	62/379	89.5399	91	77.8680
477	3389	8	2/29	21	62/379	89.7035	91	78.0316
478	3391	8	59/856	21	62/379	89.8671	91	78.1952
479	3407	8	39/566	21	62/379	90.0306	92	78.3587
480	3413	8	29/421	21	62/379	90.1942	92	78.5223
481	3433	8	23/334	21	62/379	90.3578	92	78.6859
482	3449	8	59/857	21	62/379	90.5214	92	78.8495
483	3457	8	17/247	21	62/379	90.6850	92	79.0131
484	3461	8	15/218	22	62/379	90.8486	92	79.1767
485	3463	8	13/189	22	62/379	91.0122	92	79.3403
486	3467	8	59/858	22	62/379	91.1758	92	79.5039
487	3469	8	11/160	22	62/379	91.3394	93	79.6675
488	3491	8	20/291	22	62/379	91.5029	93	79.8310
489	3499	8	9/131	22	62/379	91.6665	93	79.9946
490	3511	8	34/495	22	62/379	91.8301	93	80.1582
491	3517	8	16/233	22	62/379	91.9937	94	80.3218
492	3527	8	67/976	22	62/379	92.1573	94	80.4854
493	3529	8	7/102	22	62/379	92.3209	94	80.6490
494	3533	8	33/481	22	62/379	92.4845	94	80.8126
495	3539	8	19/277	22	62/379	92.6481	94	80.9762
496	3541	8	12/175	22	62/379	92.8116	94	81.1397
497	3547	8	17/248	22	62/379	92.9752	94	81.3033
498	3557	8	49/715	22	62/379	93.1388	94	81.4669
499	3559	8	52/759	22	62/379	93.3024	95	81.6305
500	3571	8	5/73	22	62/379	93.4660	95	81.7941
501	3581	8	48/701	22	62/379	93.6296	95	81.9577
502	3583	8	23/336	22	62/379	93.7932	95	82.1213
503	3593	8	49/716	22	62/379	93.9568	96	82.2849
504	3607	8	13/190	22	62/379	94.1204	96	82.4485
505	3613	8	29/424	22	62/379	94.2839	96	82.6120
506	3617	8	8/117	22	62/379	94.4475	96	82.7756
507	3623	8	35/512	22	62/379	94.6111	96	82.9392
508	3631	8	49/717	22	62/379	94.7747	96	83.1028
509	3637	8	11/161	22	62/379	94.9383	97	83.2664
510	3643	8	64/937	22	62/379	95.1019	97	83.4300
511	3659	8	45/659	22	62/379	95.2655	97	83.5936
512	3671	8	37/542	22	62/379	95.4291	97	83.7572
513	3673	8	23/337	22	62/379	95.5926	97	83.9207
514	3677	8	67/982	22	62/379	95.7562	97	84.0843
515	3691	8	56/821	22	62/379	95.9198	97	84.2479
516	3697	8	3/44	22	62/379	96.0834	97	84.4115
517	3701	8	3/44	22	62/379	96.2470	97	84.5751
518	3709	8	58/851	22	62/379	96.4106	97	84.7387
519	3719	8	34/499	22	62/379	96.5742	97	84.9023
520	3727	8	25/367	22	62/379	96.7378	97	85.0659
521	3733	8	19/279	22	62/379	96.9014	98	85.2294
522	3739	8	16/235	22	62/379	97.0649	98	85.3930
523	3761	8	13/191	22	62/379	97.2285	99	85.5566
524	3767	8	23/338	22	62/379	97.3921	99	85.7202
525	3769	8	10/147	22	62/379	97.5557	99	85.8838
526	3779	8	27/397	22	62/379	97.7193	99	86.0474
527	3793	8	65/956	22	62/379	97.8829	99	86.2110
528	3797	8	52/765	22	62/379	98.0465	99	86.3746
529	3803	9	7/103	23	157/994	98.2044	99	83.5541
530	3821	9	57/839	23	157/994	98.3624	99	83.7120
531	3823	9	65/957	23	157/994	98.5203	99	83.8700
532	3833	9	11/162	23	157/994	98.6783	99	84.0279
533	3847	9	26/383	23	157/994	98.8362	99	84.1859
534	3851	9	49/722	23	157/994	98.9942	99	84.3438
535	3853	9	65/958	23	157/994	99.1521	99	84.5018
536	3863	9	66/973	23	157/994	99.3100	99	84.6597
537	3877	9	63/929	23	157/994	99.4680	99	84.8177
538	3881	9	4/59	23	157/994	99.6259	99	84.9756
539	3889	9	65/959	23	157/994	99.7839	99	85.1336
540	3907	9	33/487	23	157/994	99.9418	99	85.2915
541	3911	9	67/989	23	157/994	100.0998	100	85.4494
542	3917	9	17/251	23	157/994	100.2577	100	85.6074
543	3919	9	13/192	23	157/994	100.4157	100	85.7653
544	3923	9	22/325	23	157/994	100.5736	100	85.9233

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
545	3929	9	67/990	23	157/994	100.7316	100	86.0812
546	3931	9	50/739	23	157/994	100.8895	100	86.2392
547	3943	9	37/547	23	157/994	101.0475	101	86.3971
548	3947	9	33/488	23	157/994	101.2054	101	86.5551
549	3967	9	24/355	23	157/994	101.3634	101	86.7130
550	3989	9	39/577	23	157/994	101.5213	101	86.8710
551	4001	9	5/74	23	157/994	101.6793	101	87.0289
552	4003	9	5/74	23	157/994	101.8372	101	87.1869
553	4007	9	67/992	23	157/994	101.9951	101	87.3448
554	4013	9	21/311	23	157/994	102.1531	101	87.5028
555	4019	9	59/874	23	157/994	102.3110	101	87.6607
556	4021	9	11/163	23	157/994	102.4690	101	87.8187
557	4027	9	67/993	23	157/994	102.6269	102	87.9766
558	4049	9	57/845	23	157/994	102.7849	102	88.1346
559	4051	9	64/949	23	157/994	102.9428	102	88.2925
560	4057	9	6/89	23	157/994	103.1008	102	88.4504
561	4073	9	6/89	23	157/994	103.2587	102	88.6084
562	4079	9	31/460	23	157/994	103.4167	102	88.7663
563	4091	9	19/282	23	157/994	103.5746	103	88.9243
564	4093	9	13/193	23	157/994	103.7326	103	89.0822
565	4099	9	20/297	23	157/994	103.8905	103	89.2402
566	4111	9	41/609	23	157/994	104.0485	103	89.3981
567	4127	9	7/104	23	157/994	104.2064	103	89.5561
568	4129	9	36/535	23	157/994	104.3644	103	89.7140
569	4133	9	59/877	23	157/994	104.5223	104	89.8720
570	4139	9	53/788	23	157/994	104.6803	104	90.0299
571	4153	9	39/580	23	157/994	104.8382	105	90.1879
572	4157	9	8/119	23	157/994	104.9961	105	90.3458
573	4159	9	33/491	23	157/994	105.1541	105	90.5038
574	4177	9	17/253	23	157/994	105.3120	105	90.6617
575	4201	9	35/521	23	157/994	105.4700	105	90.8197
576	4211	9	9/134	24	157/994	105.6279	105	90.9776
577	4217	9	28/417	24	157/994	105.7859	106	91.1355
578	4219	9	29/432	24	157/994	105.9438	106	91.2935
579	4229	9	10/149	24	157/994	106.1018	106	91.4514
580	4231	9	31/462	24	157/994	106.2597	106	91.6094
581	4241	9	43/641	24	157/994	106.4177	106	91.7673
582	4243	9	67/999	24	157/994	106.5756	106	91.9253
583	4253	9	58/865	24	157/994	106.7336	106	92.0832
584	4259	9	12/179	24	157/994	106.8915	106	92.2412
585	4261	9	38/567	24	157/994	107.0495	106	92.3991
586	4271	9	53/791	24	157/994	107.2074	106	92.5571
587	4273	9	14/209	24	157/994	107.3654	107	92.7150
588	4283	9	29/433	24	157/994	107.5233	107	92.8730
589	4289	9	46/687	24	157/994	107.6812	107	93.0309
590	4297	9	65/971	24	157/994	107.8392	107	93.1889
591	4327	9	17/254	24	157/994	107.9971	107	93.3468
592	4337	9	55/822	24	157/994	108.1551	107	93.5048
593	4339	9	39/583	24	157/994	108.3130	108	93.6627
594	4349	9	21/314	24	157/994	108.4710	108	93.8207
595	4357	9	45/673	24	157/994	108.6289	108	93.9786
596	4363	9	49/733	24	157/994	108.7869	108	94.1365
597	4373	9	53/793	24	157/994	108.9448	108	94.2945
598	4391	9	29/434	24	157/994	109.1028	108	94.4524
599	4397	9	65/973	24	157/994	109.2607	109	94.6104
600	4409	9	37/554	24	157/994	109.4187	109	94.7683
601	4421	9	42/629	24	157/994	109.5766	110	94.9263
602	4423	9	49/734	24	157/994	109.7346	110	95.0842
603	4441	9	58/869	24	157/994	109.8925	110	95.2422
604	4447	9	1/15	24	157/994	110.0505	110	95.4001
605	4451	9	1/15	24	157/994	110.2084	110	95.5581
606	4457	9	1/15	24	157/994	110.3664	110	95.7160
607	4463	9	1/15	24	157/994	110.5243	111	95.8740
608	4481	9	1/15	24	157/994	110.6822	111	96.0319
609	4483	9	1/15	24	157/994	110.8402	111	96.1899
610	4493	9	1/15	24	157/994	110.9981	111	96.3478
611	4507	9	1/15	24	157/994	111.1561	111	96.5058
612	4513	9	1/15	24	157/994	111.3140	111	96.6637
613	4517	9	61/916	24	157/994	111.4720	112	96.8216
614	4519	9	51/766	24	157/994	111.6299	112	96.9796
615	4523	9	44/661	24	157/994	111.7879	112	97.1375
616	4547	9	38/571	24	157/994	111.9458	112	97.2955
617	4549	9	34/511	24	157/994	112.1038	113	97.4534
618	4561	9	61/917	24	157/994	112.2617	113	97.6114
619	4567	9	55/827	24	157/994	112.4197	114	97.7693
620	4583	9	51/767	24	157/994	112.5776	114	97.9273
621	4591	9	47/707	24	157/994	112.7356	114	98.0852
622	4597	9	65/978	24	157/994	112.8935	114	98.2432
623	4603	9	61/918	24	157/994	113.0515	114	98.4011
624	4621	9	19/286	24	157/994	113.2094	114	98.5591
625	4637	9	18/271	25	157/994	113.3673	114	98.7170
626	4639	9	17/256	25	157/994	113.5253	114	98.8750
627	4643	9	16/241	25	157/994	113.6832	114	99.0329
628	4649	9	61/919	25	157/994	113.8412	114	99.1909
629	4651	9	29/437	25	157/994	113.9991	114	99.3488
630	4657	9	14/211	25	157/994	114.1571	114	99.5068
631	4663	9	40/603	25	157/994	114.3150	115	99.6647
632	4673	9	51/769	25	157/994	114.4730	115	99.8226
633	4679	9	49/739	25	157/994	114.6309	115	99.9806
634	4691	9	59/890	25	157/994	114.7889	115	100.1385
635	4703	9	34/513	25	157/994	114.9468	115	100.2965
636	4721	9	11/166	25	157/994	115.1048	115	100.4544
637	4723	9	53/800	25	157/994	115.2627	115	100.6124

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
638	4729	9	41/619	25	157/994	115.4207	115	100.7703
639	4733	9	10/151	25	157/994	115.5786	115	100.9283
640	4751	9	48/725	25	157/994	115.7366	115	101.0862
641	4759	9	28/423	25	157/994	115.8945	116	101.2442
642	4783	9	9/136	25	157/994	116.0525	116	101.4021
643	4787	9	44/665	25	157/994	116.2104	117	101.5601
644	4789	9	60/907	25	157/994	116.3683	117	101.7180
645	4793	9	25/378	25	157/994	116.5263	117	101.8760
646	4799	9	65/983	25	157/994	116.6842	117	102.0339
647	4801	9	8/121	25	157/994	116.8422	118	102.1919
648	4813	9	31/469	25	157/994	117.0001	118	102.3498
649	4817	9	15/227	25	157/994	117.1581	118	102.5077
650	4831	9	59/893	25	157/994	117.3160	118	102.6657
651	4861	9	36/545	25	157/994	117.4740	118	102.8236
652	4871	9	7/106	25	157/994	117.6319	118	102.9816
653	4877	9	7/106	25	157/994	117.7899	119	103.1395
654	4889	9	27/409	25	157/994	117.9478	119	103.2975
655	4903	9	53/803	25	157/994	118.1058	119	103.4554
656	4909	9	13/197	25	157/994	118.2637	119	103.6134
657	4919	9	19/288	25	157/994	118.4217	119	103.7713
658	4931	9	25/379	25	157/994	118.5796	119	103.9293
659	4933	9	49/743	25	157/994	118.7376	120	104.0872
660	4937	9	6/91	25	157/994	118.8955	120	104.2452
661	4943	9	59/895	25	157/994	119.0534	121	104.4031
662	4951	9	29/440	25	157/994	119.2114	121	104.5611
663	4957	9	57/865	25	157/994	119.3693	121	104.7190
664	4967	9	28/425	25	157/994	119.5273	121	104.8770
665	4969	9	11/167	25	157/994	119.6852	121	105.0349
666	4973	9	27/410	25	157/994	119.8432	121	105.1929
667	4987	9	16/243	25	157/994	120.0011	121	105.3508
668	4993	9	21/319	25	157/994	120.1591	121	105.5087
669	4999	9	31/471	25	157/994	120.3170	121	105.6667
670	5003	9	5/76	25	157/994	120.4750	121	105.8246
671	5009	9	5/76	25	157/994	120.6329	121	105.9826
672	5011	9	64/973	25	157/994	120.7909	121	106.1405
673	5021	9	34/517	25	157/994	120.9488	122	106.2985
674	5023	9	43/654	25	157/994	121.1068	122	106.4564
675	5039	9	33/502	25	157/994	121.2647	122	106.6144
676	5051	9	14/213	26	157/994	121.4227	122	106.7723
677	5059	9	55/837	26	157/994	121.5806	123	106.9303
678	5077	9	9/137	26	157/994	121.7386	123	107.0882
679	5081	9	58/883	26	157/994	121.8965	123	107.2462
680	5087	9	22/335	26	157/994	122.0544	123	107.4041
681	5099	9	13/198	26	157/994	122.2124	123	107.5621
682	5101	9	30/457	26	157/994	122.3703	123	107.7200
683	5107	9	17/259	26	157/994	122.5283	124	107.8780
684	5113	9	46/701	26	157/994	122.6862	124	108.0359
685	5119	9	62/945	26	157/994	122.8442	124	108.1938
686	5147	9	49/747	26	157/994	123.0021	124	108.3518
687	5153	9	4/61	26	157/994	123.1601	124	108.5097
688	5167	9	4/61	26	157/994	123.3180	124	108.6677
689	5171	9	63/961	26	157/994	123.4760	124	108.8256
690	5179	9	35/534	26	157/994	123.6339	124	108.9836
691	5189	9	50/763	26	157/994	123.7919	125	109.1415
692	5197	9	19/290	26	157/994	123.9498	125	109.2995
693	5209	9	64/977	26	157/994	124.1078	125	109.4574
694	5227	9	41/626	26	157/994	124.2657	125	109.6154
695	5231	9	11/168	26	157/994	124.4237	125	109.7733
696	5233	9	51/779	26	157/994	124.5816	125	109.9313
697	5237	9	18/275	26	157/994	124.7395	125	110.0892
698	5261	9	25/382	26	157/994	124.8975	125	110.2472
699	5273	9	53/810	26	157/994	125.0554	125	110.4051
700	5279	9	7/107	26	157/994	125.2134	125	110.5631
701	5281	9	45/688	26	157/994	125.3713	126	110.7210
702	5297	9	24/367	26	157/994	125.5293	126	110.8790
703	5303	9	17/260	26	157/994	125.6872	126	111.0369
704	5309	9	37/566	26	157/994	125.8452	126	111.1948
705	5323	9	10/153	26	157/994	126.0031	126	111.3528
706	5333	9	33/505	26	157/994	126.1611	126	111.5107
707	5347	9	49/750	26	157/994	126.3190	126	111.6687
708	5351	9	55/842	26	157/994	126.4770	126	111.8266
709	5381	9	16/245	26	157/994	126.6349	127	111.9846
710	5387	9	54/827	26	157/994	126.7929	127	112.1425
711	5393	9	63/965	26	157/994	126.9508	127	112.3005
712	5399	9	25/383	26	157/994	127.1088	127	112.4584
713	5407	9	65/996	26	157/994	127.2667	127	112.6164
714	5413	9	46/705	26	157/994	127.4247	127	112.7743
715	5417	9	3/46	26	157/994	127.5826	127	112.9323
716	5419	9	3/46	26	157/994	127.7405	127	113.0902
717	5431	9	3/46	26	157/994	127.8985	127	113.2482
718	5437	9	3/46	26	157/994	128.0564	127	113.4061
719	5441	9	50/767	26	157/994	128.2144	128	113.5641
720	5443	9	35/537	26	157/994	128.3723	128	113.7220
721	5449	9	26/399	26	157/994	128.5303	128	113.8799
722	5471	9	43/660	26	157/994	128.6882	128	114.0379
723	5477	9	37/568	26	157/994	128.8462	128	114.1958
724	5479	9	48/737	26	157/994	129.0041	128	114.3538
725	5483	9	14/215	26	157/994	129.1621	128	114.5117
726	5501	9	25/384	26	157/994	129.3200	128	114.6697
727	5503	9	11/169	26	157/994	129.4780	129	114.8276
728	5507	9	52/799	26	157/994	129.6359	129	114.9856
729	5519	9	19/292	27	157/994	129.7939	129	115.1435
730	5521	9	62/953	27	157/994	129.9518	129	115.3015

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
731	5527	9	8/123	27	157/994	130.1098	129	115.4594
732	5531	9	8/123	27	157/994	130.2677	129	115.6174
733	5557	9	29/446	27	157/994	130.4256	130	115.7753
734	5563	9	34/523	27	157/994	130.5836	130	115.9333
735	5569	9	13/200	27	157/994	130.7415	130	116.0912
736	5573	9	49/754	27	157/994	130.8995	130	116.2492
737	5581	9	41/631	27	157/994	131.0574	130	116.4071
738	5591	9	28/431	27	157/994	131.2154	130	116.5651
739	5623	9	48/739	27	157/994	131.3733	131	116.7230
740	5639	9	5/77	27	157/994	131.5313	131	116.8809
741	5641	9	5/77	27	157/994	131.6892	131	117.0389
742	5647	9	52/801	27	157/994	131.8472	131	117.1968
743	5651	9	59/909	27	157/994	132.0051	132	117.3548
744	5653	9	22/339	27	157/994	132.1631	132	117.5127
745	5657	9	17/262	27	157/994	132.3210	132	117.6707
746	5659	9	53/817	27	157/994	132.4790	132	117.8286
747	5669	9	12/185	27	157/994	132.6369	132	117.9866
748	5683	9	19/293	27	157/994	132.7949	132	118.1445
749	5689	9	26/401	27	157/994	132.9528	132	118.3025
750	5693	9	54/833	27	157/994	133.1108	132	118.4604
751	5701	9	7/108	27	157/994	133.2687	133	118.6184
752	5711	9	58/895	27	157/994	133.4266	133	118.7763
753	5717	9	53/818	27	157/994	133.5846	133	118.9343
754	5737	9	55/849	27	157/994	133.7425	133	118.0922
755	5741	9	41/633	27	157/994	133.9005	133	119.2502
756	5743	9	43/664	27	157/994	134.0584	133	119.4081
757	5749	9	9/139	27	157/994	134.2164	134	119.5660
758	5779	9	38/587	27	157/994	134.3743	134	119.7240
759	5783	9	20/309	27	157/994	134.5323	134	119.8819
760	5791	9	53/819	27	157/994	134.6902	134	120.0399
761	5801	9	11/170	27	157/994	134.8482	135	120.1978
762	5807	9	24/371	27	157/994	135.0061	135	120.3558
763	5813	9	13/201	27	157/994	135.1641	135	120.5137
764	5821	9	41/634	27	157/994	135.3220	135	120.6717
765	5827	9	15/232	27	157/994	135.4800	135	120.8296
766	5839	9	32/495	27	157/994	135.6379	135	120.9876
767	5843	9	17/263	27	157/994	135.7959	135	121.1455
768	5849	9	19/294	27	157/994	135.9538	135	121.3035
769	5851	9	21/325	27	157/994	136.1117	136	121.4614
770	5857	9	48/743	27	157/994	136.2697	136	121.6194
771	5861	9	27/418	27	157/994	136.4276	136	121.7773
772	5867	9	64/991	27	157/994	136.5856	136	121.9353
773	5869	9	39/604	27	157/994	136.7435	137	122.0932
774	5879	9	49/759	27	157/994	136.9015	137	122.2512
775	5881	9	2/31	27	157/994	137.0594	137	122.4091
776	5897	9	2/31	27	157/994	137.2174	137	122.5670
777	5903	9	2/31	27	157/994	137.3753	137	122.7250
778	5923	9	2/31	27	157/994	137.5333	137	122.8829
779	5927	9	2/31	27	157/994	137.6912	137	123.0409
780	5939	9	2/31	27	157/994	137.8492	137	123.1988
781	5953	9	63/977	27	157/994	138.0071	137	123.3568
782	5981	9	47/729	27	157/994	138.1651	137	123.5147
783	5987	9	37/574	27	157/994	138.3230	137	123.6727
784	6007	9	31/481	28	157/994	138.4810	137	123.8306
785	6011	9	27/419	28	157/994	138.6389	137	123.9886
786	6029	9	48/745	28	157/994	138.7969	137	124.1465
787	6037	9	21/326	28	157/994	138.9548	138	124.3045
788	6043	9	19/295	28	157/994	139.1127	138	124.4624
789	6047	9	17/264	28	157/994	139.2707	138	124.6204
790	6053	9	32/497	28	157/994	139.4286	138	124.7783
791	6067	9	15/233	28	157/994	139.5866	138	124.9363
792	6073	9	41/637	28	157/994	139.7445	138	125.0942
793	6079	9	13/202	28	157/994	139.9025	138	125.2521
794	6089	9	24/373	28	157/994	140.0604	138	125.4101
795	6091	9	57/886	28	157/994	140.2184	138	125.5680
796	6101	9	11/171	28	157/994	140.3763	138	125.7260
797	6113	9	51/793	28	157/994	140.5343	139	125.8839
798	6121	9	29/451	28	157/994	140.6922	139	126.0419
799	6131	9	9/140	28	157/994	140.8502	139	126.1998
800	6133	9	9/140	28	157/994	141.0081	139	126.3578
801	6143	9	59/918	28	157/994	141.1661	139	126.5157
802	6151	9	16/249	28	157/994	141.3240	139	126.6737
803	6163	9	39/607	28	157/994	141.4820	139	126.8316
804	6173	9	30/467	28	157/994	141.6399	139	126.9896
805	6197	9	7/109	28	157/994	141.7978	139	127.1475
806	6199	9	7/109	28	157/994	141.9558	139	127.3055
807	6203	9	47/732	28	157/994	142.1137	139	127.4634
808	6211	9	26/405	28	157/994	142.2717	139	127.6214
809	6217	9	19/296	28	157/994	142.4296	140	127.7793
810	6221	9	55/857	28	157/994	142.5876	140	127.9373
811	6229	9	12/187	28	157/994	142.7455	141	128.0952
812	6247	9	46/717	28	157/994	142.9035	141	128.2531
813	6257	9	56/873	28	157/994	143.0614	141	128.4111
814	6263	9	49/764	28	157/994	143.2194	141	128.5690
815	6269	9	37/577	28	157/994	143.3773	141	128.7270
816	6271	9	62/967	28	157/994	143.5353	141	128.8849
817	6277	9	5/78	28	157/994	143.6932	141	129.0429
818	6287	9	5/78	28	157/994	143.8512	141	129.2008
819	6299	9	48/749	28	157/994	144.0091	141	129.3588
820	6301	9	28/437	28	157/994	144.1671	141	129.5167
821	6311	9	64/999	28	157/994	144.3250	142	129.6747
822	6317	9	18/281	28	157/994	144.4830	142	129.8326
823	6323	9	57/890	28	157/994	144.6409	143	129.9906

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
824	6329	9	60/937	28	157/994	144.7988	143	130.1485
825	6337	9	21/328	28	157/994	144.9568	143	130.3065
826	6343	9	37/578	28	157/994	145.1147	143	130.4644
827	6353	9	8/125	28	157/994	145.2727	144	130.6224
828	6359	9	8/125	28	157/994	145.4306	144	130.7803
829	6361	9	62/969	28	157/994	145.5886	145	130.9382
830	6367	9	19/297	28	157/994	145.7465	145	131.0962
831	6373	9	41/641	28	157/994	145.9045	145	131.2541
832	6379	9	11/172	28	157/994	146.0624	145	131.4121
833	6389	9	36/563	28	157/994	146.2204	145	131.5700
834	6397	9	39/610	28	157/994	146.3783	145	131.7280
835	6421	9	14/219	28	157/994	146.5363	145	131.8859
836	6427	9	48/751	28	157/994	146.6942	145	132.0439
837	6449	9	37/579	28	157/994	146.8522	145	132.2018
838	6451	9	63/986	28	157/994	147.0101	145	132.3598
839	6469	9	26/407	28	157/994	147.1681	146	132.5177
840	6473	9	61/955	28	157/994	147.3260	146	132.6757
841	6481	10	38/595	29	142/929	147.4789	146	128.5487
842	6491	10	56/877	29	142/929	147.6317	146	128.7015
843	6521	10	3/47	29	142/929	147.7846	146	128.8544
844	6529	10	3/47	29	142/929	147.9374	146	129.0072
845	6547	10	3/47	29	142/929	148.0903	146	129.1601
846	6551	10	3/47	29	142/929	148.2431	146	129.3129
847	6553	10	55/862	29	142/929	148.3960	146	129.4658
848	6563	10	40/627	29	142/929	148.5488	146	129.6186
849	6569	10	31/486	29	142/929	148.7017	146	129.7715
850	6571	10	25/392	29	142/929	148.8545	146	129.9243
851	6577	10	22/345	29	142/929	149.0074	146	130.0772
852	6581	10	19/298	29	142/929	149.1602	146	130.2300
853	6599	10	16/251	29	142/929	149.3131	147	130.3829
854	6607	10	29/455	29	142/929	149.4659	147	130.5357
855	6619	10	13/204	29	142/929	149.6188	147	130.6886
856	6637	10	49/769	29	142/929	149.7716	147	130.8414
857	6653	10	56/879	29	142/929	149.9245	148	130.9943
858	6659	10	10/157	29	142/929	150.0773	148	131.1471
859	6661	10	10/157	29	142/929	150.2302	149	131.3000
860	6673	10	27/424	29	142/929	150.3830	149	131.4529
861	6679	10	17/267	29	142/929	150.5359	149	131.6057
862	6689	10	24/377	29	142/929	150.6887	149	131.7586
863	6691	10	38/597	29	142/929	150.8416	150	131.9114
864	6701	10	7/110	29	142/929	150.9945	150	132.0643
865	6703	10	7/110	29	142/929	151.1473	150	132.2171
866	6709	10	46/723	29	142/929	151.3002	150	132.3700
867	6719	10	25/393	29	142/929	151.4530	150	132.5228
868	6733	10	18/283	29	142/929	151.6059	150	132.6757
869	6737	10	29/456	29	142/929	151.7587	150	132.8285
870	6761	10	11/173	29	142/929	151.9116	150	132.9814
871	6763	10	48/755	29	142/929	152.0644	150	133.1342
872	6779	10	41/645	29	142/929	152.2173	150	133.2871
873	6781	10	15/236	29	142/929	152.3701	150	133.4399
874	6791	10	19/299	29	142/929	152.5230	150	133.5928
875	6793	10	42/661	29	142/929	152.6758	150	133.7456
876	6803	10	27/425	29	142/929	152.8287	150	133.8985
877	6823	10	35/551	29	142/929	152.9815	151	134.0513
878	6827	10	55/866	29	142/929	153.1344	151	134.2042
879	6829	10	4/63	29	142/929	153.2872	151	134.3570
880	6833	10	4/63	29	142/929	153.4401	151	134.5099
881	6841	10	4/63	29	142/929	153.5929	152	134.6627
882	6857	10	53/835	29	142/929	153.7458	152	134.8156
883	6863	10	37/583	29	142/929	153.8986	153	134.9684
884	6869	10	54/851	29	142/929	154.0515	153	135.1213
885	6871	10	21/331	29	142/929	154.2043	153	135.2742
886	6883	10	55/867	29	142/929	154.3572	153	135.4270
887	6899	10	47/741	29	142/929	154.5101	154	135.5799
888	6907	10	13/205	29	142/929	154.6629	154	135.7327
889	6911	10	61/962	29	142/929	154.8158	154	135.8856
890	6917	10	22/347	29	142/929	154.9686	154	136.0384
891	6947	10	40/631	29	142/929	155.1215	154	136.1913
892	6949	10	9/142	29	142/929	155.2743	154	136.3441
893	6959	10	59/931	29	142/929	155.4272	154	136.4970
894	6961	10	55/868	29	142/929	155.5800	154	136.6498
895	6967	10	51/805	29	142/929	155.7329	154	136.8027
896	6971	10	14/221	29	142/929	155.8857	154	136.9555
897	6977	10	19/300	29	142/929	156.0386	154	137.1084
898	6983	10	43/679	29	142/929	156.1914	154	137.2612
899	6991	10	29/458	29	142/929	156.3443	154	137.4141
900	6997	10	44/695	30	142/929	156.4971	154	137.5669
901	7001	10	5/79	30	142/929	156.6500	154	137.7198
902	7013	10	5/79	30	142/929	156.8028	154	137.8726
903	7019	10	5/79	30	142/929	156.9557	154	138.0255
904	7027	10	46/727	30	142/929	157.1085	154	138.1783
905	7039	10	31/490	30	142/929	157.2614	154	138.3312
906	7043	10	21/332	30	142/929	157.4142	154	138.4840
907	7057	10	53/838	30	142/929	157.5671	155	138.6369
908	7069	10	59/933	30	142/929	157.7199	155	138.7898
909	7079	10	38/601	30	142/929	157.8728	155	138.9426
910	7103	10	11/174	30	142/929	158.0257	155	139.0955
911	7109	10	50/791	30	142/929	158.1785	156	139.2483
912	7121	10	45/712	30	142/929	158.3314	156	139.4012
913	7127	10	57/902	30	142/929	158.4842	156	139.4450
914	7129	10	52/823	30	142/929	158.6371	156	139.7069
915	7151	10	41/649	30	142/929	158.7899	156	139.8597
916	7159	10	6/95	30	142/929	158.9428	156	140.0126

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
917	7177	10	6/95	30	142/929	159.0956	156	140.1654
918	7187	10	6/95	30	142/929	159.2485	156	140.3183
919	7193	10	37/586	30	142/929	159.4013	157	140.4711
920	7207	10	25/396	30	142/929	159.5542	157	140.6240
921	7211	10	19/301	30	142/929	159.7070	157	140.7768
922	7213	10	45/713	30	142/929	159.8599	157	140.9297
923	7219	10	13/206	30	142/929	160.0127	157	141.0825
924	7229	10	53/840	30	142/929	160.1656	157	141.2354
925	7237	10	47/745	30	142/929	160.3184	157	141.3882
926	7243	10	34/539	30	142/929	160.4713	157	141.5411
927	7247	10	7/111	30	142/929	160.6241	157	141.6939
928	7253	10	7/111	30	142/929	160.7770	157	141.8468
929	7283	10	57/904	30	142/929	160.9298	158	141.9996
930	7297	10	29/460	30	142/929	161.0827	158	142.1525
931	7307	10	22/349	30	142/929	161.2355	158	142.3054
932	7309	10	15/238	30	142/929	161.3884	158	142.4582
933	7321	10	38/603	30	142/929	161.5413	158	142.6111
934	7331	10	31/492	30	142/929	161.6941	158	142.7639
935	7333	10	55/873	30	142/929	161.8470	158	142.9168
936	7349	10	8/127	30	142/929	161.9998	158	143.0696
937	7351	10	57/905	30	142/929	162.1527	159	143.2225
938	7369	10	58/921	30	142/929	162.3055	159	143.3753
939	7393	10	59/937	30	142/929	162.4584	159	143.5282
940	7411	10	43/683	30	142/929	162.6112	159	143.6810
941	7417	10	35/556	30	142/929	162.7641	160	143.8339
942	7433	10	9/143	30	142/929	162.9169	160	143.9867
943	7451	10	9/143	30	142/929	163.0698	160	144.1396
944	7457	10	37/588	30	142/929	163.2226	160	144.2924
945	7459	10	19/302	30	142/929	163.3755	160	144.4453
946	7477	10	29/461	30	142/929	163.5283	160	144.5981
947	7481	10	59/938	30	142/929	163.6812	161	144.7510
948	7487	10	10/159	30	142/929	163.8340	161	144.9038
949	7489	10	41/652	30	142/929	163.9869	161	145.0567
950	7499	10	21/334	30	142/929	164.1397	161	145.2095
951	7507	10	43/684	30	142/929	164.2926	161	145.3624
952	7517	10	11/175	30	142/929	164.4454	161	145.5152
953	7523	10	45/716	30	142/929	164.5983	162	145.6681
954	7529	10	23/366	30	142/929	164.7511	162	145.8210
955	7537	10	59/939	30	142/929	164.9040	162	145.9738
956	7541	10	12/191	30	142/929	165.0569	162	146.1267
957	7547	10	25/398	30	142/929	165.2097	162	146.2795
958	7549	10	51/812	30	142/929	165.3626	162	146.4324
959	7559	10	13/207	30	142/929	165.5154	162	146.5852
960	7561	10	27/430	30	142/929	165.6683	162	146.7381
961	7573	11	14/223	31	29/195	165.8170	162	142.9209
962	7577	11	43/685	31	29/195	165.9657	162	143.0696
963	7583	11	59/940	31	29/195	166.1144	162	143.2183
964	7589	11	61/972	31	29/195	166.2631	162	143.3671
965	7591	11	47/749	31	29/195	166.4119	162	143.5158
966	7603	11	49/781	31	29/195	166.5606	162	143.6645
967	7607	11	17/271	31	29/195	166.7093	163	143.8132
968	7621	11	35/558	31	29/195	166.8580	163	143.9619
969	7639	11	18/287	31	29/195	167.0067	163	144.1107
970	7643	11	19/303	31	29/195	167.1555	163	144.2594
971	7649	11	59/941	31	29/195	167.3042	164	144.4081
972	7669	11	41/654	31	29/195	167.4529	164	144.5568
973	7673	11	43/686	31	29/195	167.6016	164	144.7055
974	7681	11	45/718	31	29/195	167.7504	164	144.8543
975	7687	11	47/750	31	29/195	167.8991	164	145.0030
976	7691	11	49/782	31	29/195	168.0478	164	145.1517
977	7699	11	26/415	31	29/195	168.1965	165	145.3004
978	7703	11	55/878	31	29/195	168.3452	165	145.4491
979	7717	11	29/463	31	29/195	168.4940	165	145.5979
980	7723	11	31/495	31	29/195	168.6427	165	145.7466
981	7727	11	33/527	31	29/195	168.7914	165	145.8953
982	7741	11	36/575	31	29/195	168.9401	165	146.0440
983	7753	11	38/607	31	29/195	169.0888	166	146.1928
984	7757	11	42/671	31	29/195	169.2376	166	146.3415
985	7759	11	45/719	31	29/195	169.3863	166	146.4902
986	7789	11	50/799	31	29/195	169.5350	166	146.6389
987	7793	11	56/895	31	29/195	169.6837	166	146.7876
988	7817	11	1/16	31	29/195	169.8324	166	146.9364
989	7823	11	1/16	31	29/195	169.9812	166	147.0851
990	7829	11	1/16	31	29/195	170.1299	166	147.2338
991	7841	11	1/16	31	29/195	170.2786	167	147.3825
992	7853	11	1/16	31	29/195	170.4273	167	147.5312
993	7867	11	1/16	31	29/195	170.5761	167	147.6800
994	7873	11	1/16	31	29/195	170.7248	167	147.8287
995	7877	11	1/16	31	29/195	170.8735	167	147.9774
996	7879	11	1/16	31	29/195	171.0222	167	148.1261
997	7883	11	1/16	31	29/195	171.1709	168	148.2748
998	7901	11	1/16	31	29/195	171.3197	168	148.4236
999	7907	11	1/16	31	29/195	171.4684	168	148.5723
1000	7919	11	1/16	31	29/195	171.6171	168	148.7210
1001	7927	11	1/16	31	29/195	171.7658	168	148.8697
1002	7933	11	1/16	31	29/195	171.9145	169	149.0185
1003	7937	11	1/16	31	29/195	172.0633	169	149.1672
1004	7949	11	60/961	31	29/195	172.2120	169	149.3159
1005	7951	11	54/865	31	29/195	172.3607	169	149.4646
1006	7963	11	48/769	31	29/195	172.5094	169	149.6133
1007	7993	11	44/705	31	29/195	172.6581	169	149.7621
1008	8009	11	41/657	31	29/195	172.8069	169	149.9108
1009	8011	11	38/609	31	29/195	172.9556	169	150.0595

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
1010	8017	11	35/561	31	29/195	173.1043	169	150.2082
1011	8039	11	33/529	31	29/195	173.2530	169	150.3569
1012	8053	11	61/978	31	29/195	173.4018	169	150.5057
1013	8059	11	29/465	31	29/195	173.5505	170	150.6544
1014	8069	11	27/433	31	29/195	173.6992	170	150.8031
1015	8081	11	26/417	31	29/195	173.8479	170	150.9518
1016	8087	11	49/786	31	29/195	173.9966	170	151.1005
1017	8089	11	47/754	31	29/195	174.1454	170	151.2493
1018	8093	11	45/722	31	29/195	174.2941	170	151.3980
1019	8101	11	43/690	31	29/195	174.4428	171	151.5467
1020	8111	11	62/995	31	29/195	174.5915	171	151.6954
1021	8117	11	20/321	31	29/195	174.7402	172	151.8442
1022	8123	11	19/305	31	29/195	174.8890	172	151.9929
1023	8147	11	55/883	31	29/195	175.0377	172	152.1416
1024	8161	11	53/851	32	29/195	175.1864	172	152.2903
1025	8167	11	17/273	32	29/195	175.3351	172	152.4390
1026	8171	11	33/530	32	29/195	175.4838	172	152.5878
1027	8179	11	16/257	32	29/195	175.6326	172	152.7365
1028	8191	11	31/498	32	29/195	175.7813	172	152.8852
1029	8209	11	15/241	32	29/195	175.9300	172	153.0339
1030	8219	11	44/707	32	29/195	176.0787	172	153.1826
1031	8221	11	57/916	32	29/195	176.2275	173	153.3314
1032	8231	11	14/225	32	29/195	176.3762	173	153.4801
1033	8233	11	27/434	32	29/195	176.5249	174	153.6288
1034	8237	11	13/209	32	29/195	176.6736	174	153.7775
1035	8243	11	13/209	32	29/195	176.8223	174	153.9262
1036	8263	11	25/402	32	29/195	176.9711	174	154.0750
1037	8269	11	61/981	32	29/195	177.1198	174	154.2237
1038	8273	11	12/193	32	29/195	177.2685	174	154.3724
1039	8287	11	35/563	32	29/195	177.4172	175	154.5211
1040	8291	11	57/917	32	29/195	177.5659	175	154.6699
1041	8293	11	11/177	32	29/195	177.7147	175	154.8186
1042	8297	11	11/177	32	29/195	177.8634	175	154.9673
1043	8311	11	32/515	32	29/195	178.0121	175	155.1160
1044	8317	11	21/338	32	29/195	178.1608	175	155.2647
1045	8329	11	41/660	32	29/195	178.3095	175	155.4135
1046	8353	11	10/161	32	29/195	178.4583	175	155.5622
1047	8363	11	10/161	32	29/195	178.6070	175	155.7109
1048	8369	11	29/467	32	29/195	178.7557	175	155.8596
1049	8377	11	19/306	32	29/195	178.9044	176	156.0083
1050	8387	11	28/451	32	29/195	179.0532	176	156.1571
1051	8389	11	46/741	32	29/195	179.2019	177	156.3058
1052	8419	11	9/145	32	29/195	179.3506	177	156.4545
1053	8423	11	62/999	32	29/195	179.4993	177	156.6032
1054	8429	11	61/983	32	29/195	179.6480	177	156.7519
1055	8431	11	60/967	32	29/195	179.7968	177	156.9007
1056	8443	11	59/951	32	29/195	179.9455	177	157.0494
1057	8447	11	25/403	32	29/195	180.0942	177	157.1981
1058	8461	11	49/790	32	29/195	180.2429	177	157.3468
1059	8467	11	8/129	32	29/195	180.3916	177	157.4956
1060	8501	11	8/129	32	29/195	180.5404	177	157.6443
1061	8513	11	39/629	32	29/195	180.6891	178	157.7930
1062	8521	11	54/871	32	29/195	180.8378	178	157.9417
1063	8527	11	38/613	32	29/195	180.9865	179	158.0904
1064	8537	11	15/242	32	29/195	181.1352	179	158.2392
1065	8539	11	59/952	32	29/195	181.2840	179	158.3879
1066	8543	11	51/823	32	29/195	181.4327	179	158.5366
1067	8563	11	43/694	32	29/195	181.5814	179	158.6853
1068	8573	11	7/113	32	29/195	181.7301	179	158.8340
1069	8581	11	7/113	32	29/195	181.8789	180	158.9828
1070	8597	11	7/113	32	29/195	182.0276	180	159.1315
1071	8599	11	34/549	32	29/195	182.1763	180	159.2802
1072	8609	11	47/759	32	29/195	182.3250	180	159.4289
1073	8623	11	20/323	32	29/195	182.4737	180	159.5776
1074	8627	11	59/953	32	29/195	182.6225	180	159.7264
1075	8629	11	13/210	32	29/195	182.7712	180	159.8751
1076	8641	11	32/517	32	29/195	182.9199	180	160.0238
1077	8647	11	19/307	32	29/195	183.0686	180	160.1725
1078	8663	11	25/404	32	29/195	183.2173	180	160.3213
1079	8669	11	37/598	32	29/195	183.3661	180	160.4700
1080	8677	11	55/889	32	29/195	183.5148	180	160.6187
1081	8681	11	6/97	32	29/195	183.6635	180	160.7674
1082	8689	11	6/97	32	29/195	183.8122	180	160.9161
1083	8693	11	6/97	32	29/195	183.9609	180	161.0649
1084	8699	11	35/566	32	29/195	184.1097	180	161.2136
1085	8707	11	52/841	32	29/195	184.2584	180	161.3623
1086	8713	11	40/647	32	29/195	184.4071	180	161.5110
1087	8719	11	17/275	32	29/195	184.5558	181	161.6597
1088	8731	11	28/453	32	29/195	184.7046	181	161.8085
1089	8737	11	61/987	33	29/195	184.8533	181	161.9572
1090	8741	11	11/178	33	29/195	185.0020	181	162.1059
1091	8747	11	38/615	33	29/195	185.1507	182	162.2546
1092	8753	11	43/696	33	29/195	185.2994	182	162.4033
1093	8761	11	16/259	33	29/195	185.4482	183	162.5521
1094	8779	11	58/939	33	29/195	185.5969	183	162.7008
1095	8783	11	47/761	33	29/195	185.7456	183	162.8495
1096	8803	11	31/502	33	29/195	185.8943	183	162.9982
1097	8807	11	41/664	33	29/195	186.0430	184	163.1470
1098	8819	11	61/988	33	29/195	186.1918	184	163.2957
1099	8821	11	5/81	33	29/195	186.3405	184	163.4444
1100	8831	11	5/81	33	29/195	186.4892	184	163.5931
1101	8837	11	5/81	33	29/195	186.6379	184	163.7418
1102	8839	11	49/794	33	29/195	186.7866	184	163.8906

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
1103	8849	11	34/551	33	29/195	186.9354	185	164.0393
1104	8861	11	24/389	33	29/195	187.0841	185	164.1880
1105	8863	11	19/308	33	29/195	187.2328	185	164.3367
1106	8867	11	52/843	33	29/195	187.3815	185	164.4854
1107	8887	11	14/227	33	29/195	187.5302	185	164.6342
1108	8893	11	14/227	33	29/195	187.6790	185	164.7829
1109	8923	11	23/373	33	29/195	187.8277	186	164.9316
1110	8929	11	32/519	33	29/195	187.9764	186	1650803
1111	8933	11	9/146	33	29/195	188.1251	186	165.2290
1112	8941	11	9/146	33	29/195	188.2739	186	165.3778
1113	8951	11	49/795	33	29/195	188.4226	186	165.5265
1114	8963	11	31/503	33	29/195	188.5713	186	165.6752
1115	8969	11	57/925	33	29/195	188.7200	186	165.8239
1116	8971	11	13/211	33	29/195	188.8687	186	165.9726
1117	8999	11	13/211	33	29/195	189.0175	187	166.1214
1118	9001	11	30/487	33	29/195	189.1662	187	166.2701
1119	9007	11	17/276	33	29/195	189.3149	187	166.4188
1120	9011	11	38/617	33	29/195	189.4636	187	166.5675
1121	9013	11	21/341	33	29/195	189.6123	187	166.7163
1122	9029	11	54/877	33	29/195	189.7611	187	166.8650
1123	9041	11	33/536	33	29/195	189.9098	188	167.0137
1124	9043	11	41/666	33	29/195	190.0585	188	167.1624
1125	9049	11	61/991	33	29/195	190.2072	188	167.3111
1126	9059	11	4/65	33	29/195	190.3559	188	167.4599
1127	9067	11	4/65	33	29/195	190.5047	188	167.6086
1128	9091	11	4/65	33	29/195	190.6534	188	167.7573
1129	9103	11	4/65	33	29/195	190.8021	189	167.9060
1130	9109	11	51/829	33	29/195	190.9508	189	168.0547
1131	9127	11	39/634	33	29/195	191.0996	189	168.2035
1132	9133	11	58/943	33	29/195	191.2483	189	168.3522
1133	9137	11	50/813	33	29/195	191.3970	189	168.5009
1134	9151	11	42/683	33	29/195	191.5457	189	168.6496
1135	9157	11	19/309	33	29/195	191.6944	189	168.7983
1136	9161	11	49/797	33	29/195	191.8432	189	168.9471
1137	9173	11	15/244	33	29/195	191.9919	189	169.0958
1138	9181	11	26/423	33	29/195	192.1406	189	169.2445
1139	9187	11	48/781	33	29/195	192.2893	189	169.3932
1140	9199	11	11/179	33	29/195	192.4380	189	169.5420
1141	9203	11	51/830	33	29/195	192.5868	189	169.6907
1142	9209	11	29/472	33	29/195	192.7355	189	169.8394
1143	9221	11	18/293	33	29/195	192.8842	189	169.9881
1144	9227	11	25/407	33	29/195	193.0329	189	170.1368
1145	9239	11	32/521	33	29/195	193.1816	189	170.2856
1146	9241	11	60/977	33	29/195	193.3304	189	170.4343
1147	9257	11	7/114	33	29/195	193.4791	189	170.5830
1148	9277	11	7/114	33	29/195	193.6278	189	170.7317
1149	9281	11	52/847	33	29/195	193.7765	189	170.8804
1150	9283	11	31/505	33	29/195	193.9253	189	171.0292
1151	9293	11	24/391	33	29/195	194.0740	190	171.1779
1152	9311	11	17/277	33	29/195	194.2227	190	171.3266
1153	9319	11	44/717	33	29/195	194.3714	191	171.4753
1154	9323	11	37/603	33	29/195	194.5201	191	171.6240
1155	9337	11	10/163	33	29/195	194.6689	191	171.7728
1156	9341	11	10/163	34	29/195	194.8176	191	171.9215
1157	9343	11	43/701	34	29/195	194.9663	191	172.0702
1158	9349	11	23/375	34	29/195	195.1150	191	172.2189
1159	9371	11	49/799	34	29/195	195.2637	191	172.3677
1160	9377	11	13/212	34	29/195	195.4125	191	172.5164
1161	9391	11	42/685	34	29/195	195.5612	191	172.6651
1162	9397	11	45/734	34	29/195	195.7099	191	172.8138
1163	9403	11	16/261	34	29/195	195.8586	192	172.9625
1164	9413	11	54/881	34	29/195	196.0073	192	173.1113
1165	9419	11	19/310	34	29/195	196.1561	192	173.2600
1166	9421	11	22/359	34	29/195	196.3048	192	173.4087
1167	9431	11	25/408	34	29/195	196.4535	192	173.5574
1168	9433	11	28/457	34	29/195	196.6022	192	173.7061
1169	9437	11	34/555	34	29/195	196.7510	192	173.8549
1170	9439	11	40/653	34	29/195	196.8997	192	174.0036
1171	9461	11	52/849	34	29/195	197.0484	193	174.1523
1172	9463	11	3/49	34	29/195	197.1971	193	174.3010
1173	9467	11	3/49	34	29/195	197.3458	193	174.4497
1174	9473	11	3/49	34	29/195	197.4946	193	174.5985
1175	9479	11	3/49	34	29/195	197.6433	193	174.7472
1176	9491	11	3/49	34	29/195	197.7920	193	174.8959
1177	9497	11	3/49	34	29/195	197.9407	193	175.0446
1178	9511	11	59/964	34	29/195	198.0894	193	175.1934
1179	9521	11	47/768	34	29/195	198.2382	193	175.3421
1180	9533	11	38/621	34	29/195	198.3869	193	175.4908
1181	9539	11	32/523	34	29/195	198.5356	194	175.6395
1182	9547	11	26/425	34	29/195	198.6843	194	175.7882
1183	9551	11	23/376	34	29/195	198.8330	194	175.9370
1184	9587	11	43/703	34	29/195	198.9818	194	176.0857
1185	9601	11	57/932	34	29/195	199.1305	194	176.2344
1186	9613	11	17/278	34	29/195	199.2792	194	176.3831
1187	9619	11	48/785	34	29/195	199.4279	195	176.5318
1188	9623	11	59/965	34	29/195	199.5767	195	176.6806
1189	9629	11	14/229	34	29/195	199.7254	195	176.8293
1190	9631	11	39/638	34	29/195	199.8741	195	176.9780
1191	9643	11	36/589	34	29/195	200.0228	195	177.1267
1192	9649	11	11/180	34	29/195	200.1715	195	177.2754
1193	9661	11	11/180	34	29/195	200.3203	196	177.4242
1194	9677	11	41/671	34	29/195	200.4690	196	177.5729
1195	9679	11	19/311	34	29/195	200.6177	196	177.7216

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
1196	9689	11	46/753	34	29/195	200.7664	196	177.8703
1197	9697	11	35/573	34	29/195	200.9151	196	178.0191
1198	9719	11	59/966	34	29/195	201.0639	196	178.1678
1199	9721	11	8/131	34	29/195	201.2126	196	178.3165
1200	9733	11	8/131	34	29/195	201.3613	196	178.4652
1201	9739	11	45/737	34	29/195	201.5100	197	178.6139
1202	9743	11	29/475	34	29/195	201.6587	197	178.7627
1203	9749	11	21/344	34	29/195	201.8075	197	178.9114
1204	9767	11	47/770	34	29/195	201.9562	197	179.0601
1205	9769	11	13/213	34	29/195	202.1049	197	179.2088
1206	9781	11	44/721	34	29/195	202.2536	197	179.3575
1207	9787	11	49/803	34	29/195	202.4024	197	179.5063
1208	9791	11	59/967	34	29/195	202.5511	197	179.6550
1209	9803	11	23/377	34	29/195	202.6998	197	179.8037
1210	9811	11	28/459	34	29/195	202.8485	197	179.9524
1211	9817	11	38/623	34	29/195	202.9972	197	180.1011
1212	9829	11	53/869	34	29/195	203.1460	197	180.2499
1213	9833	11	5/82	34	29/195	203.2947	198	180.3986
1214	9839	11	5/82	34	29/195	203.4434	198	180.5473
1215	9851	11	5/82	34	29/195	203.5921	198	180.6960
1216	9857	11	5/82	34	29/195	203.7408	198	180.8448
1217	9859	11	42/689	34	29/195	203.8896	199	180.9935
1218	9871	11	32/525	34	29/195	204.0383	199	181.1422
1219	9883	11	49/804	34	29/195	204.1870	199	181.2909
1220	9887	11	22/361	34	29/195	204.3357	199	181.4396
1221	9901	11	17/279	34	29/195	204.4844	199	181.5884
1222	9907	11	46/755	34	29/195	204.6332	199	181.7371
1223	9923	11	41/673	34	29/195	204.7819	200	181.8858
1224	9929	11	12/197	34	29/195	204.9306	200	182.0345
1225	9931	11	12/197	35	29/195	205.0793	200	182.1832
1226	9941	11	31/509	35	29/195	205.2281	200	182.3320
1227	9949	11	19/312	35	29/195	205.3768	200	182.4807
1228	9967	11	26/427	35	29/195	205.5255	200	182.6294
1229	9973	11	26/427	35	29/195	205.6742	201	182.7781
1230	10007	11	33/542	35	29/195	205.8229	201	182.9268
1231	10009	11	33/542	35	29/195	205.9717	202	183.0756
1232	10037	11	54/887	35	29/195	206.1204	202	183.2243
1233	10039	11	54/887	35	29/195	206.2691	202	183.3730
1234	10061	11	7/115	35	29/195	206.4178	202	183.5217
1235	10067	11	7/115	35	29/195	206.5665	202	183.6705
1236	10069	11	7/115	35	29/195	206.7153	202	183.8192
1237	10079	11	7/115	35	29/195	206.8640	203	183.9679
1238	10091	11	58/953	35	29/195	207.0127	203	184.1166
1239	10093	11	58/953	35	29/195	207.1614	203	184.2653
1240	10099	11	37/608	35	29/195	207.3101	203	184.4141
1241	10103	11	37/608	35	29/195	207.4589	203	184.5628
1242	10111	11	23/378	35	29/195	207.6076	203	184.7115
1243	10133	11	23/378	35	29/195	207.7563	203	184.8602
1244	10139	11	39/641	35	29/195	207.9050	203	185.0089
1245	10141	11	39/641	35	29/195	208.0537	203	185.1577
1246	10151	11	16/263	35	29/195	208.2025	203	185.3064
1247	10159	11	16/263	35	29/195	208.3512	203	185.4451
1248	10163	11	41/674	35	29/195	208.4999	203	185.6038
1249	10169	11	41/674	35	29/195	208.6486	204	185.7525
1250	10177	11	59/970	35	29/195	208.7974	204	185.9013
1251	10181	11	59/970	35	29/195	208.9461	204	186.0500
1252	10193	11	52/855	35	29/195	209.0948	204	186.1987
1253	10211	11	52/855	35	29/195	209.2435	204	186.3474
1254	10223	11	9/148	35	29/195	209.3922	204	186.4961
1255	10243	11	9/148	35	29/195	209.5410	204	186.6449
1256	10247	11	9/148	35	29/195	209.6897	204	186.7936
1257	10253	11	9/148	35	29/195	209.8384	204	186.9423
1258	10259	11	38/625	35	29/195	209.9871	204	187.0910
1259	10267	11	38/625	35	29/195	210.1358	205	187.2398
1260	10271	11	49/806	35	29/195	210.2846	205	187.3885
1261	10273	11	49/806	35	29/195	210.4333	205	187.5372
1262	10289	11	20/329	35	29/195	210.5820	205	187.6859
1263	10301	11	20/329	35	29/195	210.7307	205	187.8346
1264	10303	11	31/510	35	29/195	210.8794	205	187.9834
1265	10313	11	31/510	35	29/195	211.0282	205	188.1321
1266	10321	11	11/181	35	29/195	211.1769	205	188.2808
1267	10331	11	11/181	35	29/195	211.3256	205	188.4295
1268	10333	11	11/181	35	29/195	211.4743	205	188.5782
1269	10337	11	11/181	35	29/195	211.6231	205	188.7270
1270	10343	11	46/757	35	29/195	211.7718	205	188.8757
1271	10357	11	46/757	35	29/195	211.9205	205	189.0244
1272	10369	11	24/395	35	29/195	212.0692	205	189.1731
1273	10391	11	24/395	35	29/195	212.2179	205	189.3218
1274	10399	11	50/823	35	29/195	212.3667	205	189.4706
1275	10427	11	50/823	35	29/195	212.5154	205	189.6193
1276	10429	11	13/214	35	29/195	212.6641	205	189.7680
1277	10433	11	13/214	35	29/195	212.8128	206	189.9167
1278	10453	11	54/889	35	29/195	212.9615	206	190.0655
1279	10457	11	54/889	35	29/195	213.1103	207	190.2142
1280	10459	11	28/461	35	29/195	213.2590	207	190.3629
1281	10463	11	28/461	35	29/195	213.4077	207	190.5116
1282	10477	11	15/247	35	29/195	213.5564	207	190.6603
1283	10487	11	15/247	35	29/195	213.7051	208	190.8091
1284	10499	11	47/774	35	29/195	213.8539	208	190.9578
1285	10501	11	47/774	35	29/195	214.0026	208	191.1065
1286	10513	11	49/807	35	29/195	214.1513	208	191.2552
1287	10529	11	49/807	35	29/195	214.3000	208	191.4039
1288	10531	11	17/280	35	29/195	214.4488	208	191.5527

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
1289	10559	11	17/280	35	29/195	214.5975	209	191.7014
1290	10567	11	36/593	35	29/195	214.7462	209	191.8501
1291	10589	11	36/593	35	29/195	214.8949	210	191.9988
1292	10597	11	19/313	35	29/195	215.0436	210	192.1475
1293	10601	11	19/313	35	29/195	215.1924	210	192.2963
1294	10607	11	21/346	35	29/195	215.3411	210	192.4450
1295	10613	11	21/346	35	29/195	215.4898	210	192.5937
1296	10627	11	44/725	36	29/195	215.6385	210	192.7424
1297	10631	11	44/725	36	29/195	215.7872	211	192.8912
1298	10639	11	23/379	36	29/195	215.9360	211	193.0399
1299	10651	11	23/379	36	29/195	216.0847	211	193.1886
1300	10657	11	25/412	36	29/195	216.2334	211	193.3373
1301	10663	11	25/412	36	29/195	216.3821	212	193.4860
1302	10667	11	27/445	36	29/195	216.5308	212	193.6348
1303	10687	11	27/445	36	29/195	216.6796	213	193.7835
1304	10691	11	60/989	36	29/195	216.8283	213	193.9322
1305	10709	11	60/989	36	29/195	216.9770	213	194.0809
1306	10711	11	33/544	36	29/195	217.1257	213	194.2296
1307	10723	11	33/544	36	29/195	217.2745	214	194.3784
1308	10729	11	37/610	36	29/195	217.4232	214	194.5271
1309	10733	11	37/610	36	29/195	217.5719	214	194.6758
1310	10739	11	41/676	36	29/195	217.7206	214	194.8245
1311	10753	11	41/676	36	29/195	217.8693	214	194.9732
1312	10771	11	47/775	36	29/195	218.0181	214	195.1220
1313	10781	11	47/775	36	29/195	218.1668	214	195.2707
1314	10789	11	55/907	36	29/195	218.3155	214	195.4194
1315	10799	11	55/907	36	29/195	218.4642	214	195.5681
1316	10831	11	2/33	36	29/195	218.6129	214	195.7169
1317	10837	11	2/33	36	29/195	218.7617	214	195.8656
1318	10847	11	2/33	36	29/195	218.9104	214	196.0143
1319	10853	11	2/33	36	29/195	219.0591	215	196.1630
1320	10859	11	2/33	36	29/195	219.2078	215	196.3117
1321	10861	11	2/33	36	29/195	219.3565	216	196.4605
1322	10867	11	2/33	36	29/195	219.5053	216	196.6092
1323	10883	11	2/33	36	29/195	219.6540	216	196.7579
1324	10889	11	2/33	36	29/195	219.8027	216	196.9066
1325	10891	11	2/33	36	29/195	219.9514	216	197.0553
1326	10903	11	2/33	36	29/195	220.1002	216	197.2041
1327	10909	11	2/33	36	29/195	220.2489	217	197.3528
1328	10937	11	2/33	36	29/195	220.3976	217	197.5015
1329	10939	11	2/33	36	29/195	220.5463	217	197.6502
1330	10949	11	2/33	36	29/195	220.6950	217	197.7989
1331	10957	11	2/33	36	29/195	220.8438	217	197.9477
1332	10973	11	2/33	36	29/195	220.9925	217	198.0964
1333	10979	11	2/33	36	29/195	221.1412	217	198.2451
1334	10987	11	2/33	36	29/195	221.2899	217	198.3938
1335	10993	11	2/33	36	29/195	221.4386	217	198.5426
1336	11003	11	2/33	36	29/195	221.5874	217	198.6913
1337	11027	11	2/33	36	29/195	221.7361	217	198.8400
1338	11047	11	55/908	36	29/195	221.8848	217	198.9887
1339	11057	11	55/908	36	29/195	222.0335	217	199.1374
1340	11059	11	47/776	36	29/195	222.1822	217	199.2862
1341	11069	11	47/776	36	29/195	222.3310	217	199.4349
1342	11071	11	41/677	36	29/195	222.4797	217	199.5836
1343	11083	11	41/677	36	29/195	222.6284	217	199.7323
1344	11087	11	37/611	36	29/195	222.7771	217	199.8810
1345	11093	11	37/611	36	29/195	222.9259	217	200.0298
1346	11113	11	33/545	36	29/195	223.0746	217	200.1785
1347	11117	11	33/545	36	29/195	223.2233	217	200.3272
1348	11119	11	60/991	36	29/195	223.3720	217	200.4759
1349	11131	11	31/512	36	29/195	223.5207	217	200.6246
1350	11149	11	56/925	36	29/195	223.6695	217	200.7734
1351	11159	11	56/925	36	29/195	223.8182	217	200.9221
1352	11161	11	25/413	36	29/195	223.9669	217	201.0708
1353	11171	11	52/859	36	29/195	224.1156	217	201.2195
1354	11173	11	48/793	36	29/195	224.2643	217	201.3683
1355	11177	11	48/793	36	29/195	224.4131	217	201.5170
1356	11197	11	44/727	36	29/195	224.5618	217	201.6657
1357	11213	11	23/380	36	29/195	224.7105	217	201.8144
1358	11239	11	21/347	36	29/195	224.8592	217	201.9631
1359	11243	11	21/347	36	29/195	225.0079	217	202.1119
1360	11251	11	59/975	36	29/195	225.1567	217	202.2606
1361	11257	11	59/975	36	29/195	225.3054	218	202.4093
1362	11261	11	19/314	36	29/195	225.4541	218	202.5580
1363	11273	11	19/314	36	29/195	225.6028	218	202.7067
1364	11279	11	53/876	36	29/195	225.7515	218	202.8555
1365	11287	11	53/876	36	29/195	225.9003	218	203.0042
1366	11299	11	17/281	36	29/195	226.0490	218	203.1529
1367	11311	11	17/281	36	29/195	226.1977	219	203.3016
1368	11317	11	32/529	36	29/195	226.3464	219	203.4503
1369	11321	12	32/529	37	139/958	226.4915	219	198.6332
1370	11329	12	15/248	37	139/958	226.6366	219	198.7783
1371	11351	12	15/248	37	139/958	226.7817	219	198.9234
1372	11353	12	15/248	37	139/958	226.9268	219	199.0685
1373	11369	12	15/248	37	139/958	227.0719	220	199.2136
1374	11383	12	28/463	37	139/958	227.2170	220	199.3587
1375	11393	12	28/463	37	139/958	227.3621	220	199.5038
1376	11399	12	54/893	37	139/958	227.5072	220	199.6489
1377	11411	12	54/893	37	139/958	227.6523	220	199.7940
1378	11423	12	13/215	37	139/958	227.7974	220	199.9391
1379	11437	12	13/215	37	139/958	227.9425	220	200.0842
1380	11443	12	50/827	37	139/958	228.0876	220	200.2293
1381	11447	12	50/827	37	139/958	228.2327	221	200.3744

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
1382	11467	12	24/397	37	139/958	228.3777	221	200.5195
1383	11471	12	24/397	37	139/958	228.5228	221	200.6645
1384	11483	12	35/579	37	139/958	228.6679	221	200.8096
1385	11489	12	35/579	37	139/958	228.8130	221	200.9547
1386	11491	12	11/182	37	139/958	228.9581	221	201.0998
1387	11497	12	11/182	37	139/958	229.1032	221	201.2449
1388	11503	12	11/182	37	139/958	229.2483	221	201.3900
1389	11519	12	11/182	37	139/958	229.3934	221	201.5351
1390	11527	12	53/877	37	139/958	229.5385	221	201.6802
1391	11549	12	53/877	37	139/958	229.6836	221	201.8253
1392	11551	12	31/513	37	139/958	229.8287	221	201.9704
1393	11579	12	31/513	37	139/958	229.9738	221	202.1155
1394	11587	12	20/331	37	139/958	230.1189	221	202.2606
1395	11593	12	20/331	37	139/958	230.2640	221	202.4057
1396	11597	12	29/480	37	139/958	230.4091	221	202.5508
1397	11617	12	29/480	37	139/958	230.5542	221	202.6959
1398	11621	12	38/629	37	139/958	230.6992	221	202.8410
1399	11633	12	38/629	37	139/958	230.8443	222	202.9860
1400	11657	12	9/149	37	139/958	230.9894	222	203.1311
1401	11677	12	9/149	37	139/958	231.1345	222	203.2762
1402	11681	12	9/149	37	139/958	231.2796	222	203.4213
1403	11689	12	9/149	37	139/958	231.4247	222	203.5664
1404	11699	12	9/149	37	139/958	231.5698	222	203.7115
1405	11701	12	9/149	37	139/958	231.7149	222	203.8566
1406	11717	12	34/563	37	139/958	231.8600	222	204.0017
1407	11719	12	43/712	37	139/958	232.0051	222	204.1468
1408	11731	12	25/414	37	139/958	232.1502	222	204.2919
1409	11743	12	25/414	37	139/958	232.2953	223	204.4370
1410	11777	12	57/944	37	139/958	232.4404	223	204.5821
1411	11779	12	57/944	37	139/958	232.5855	223	204.7272
1412	11783	12	16/265	37	139/958	232.7306	223	204.8723
1413	11789	12	16/265	37	139/958	232.8756	223	205.0174
1414	11801	12	39/646	37	139/958	233.0207	223	205.1624
1415	11807	12	39/646	37	139/958	233.1658	223	205.3075
1416	11813	12	23/381	37	139/958	233.3109	223	205.4526
1417	11821	12	23/381	37	139/958	233.4560	223	205.5977
1418	11827	12	30/497	37	139/958	233.6011	223	205.7428
1419	11831	12	30/497	37	139/958	233.7462	223	205.8879
1420	11833	12	44/729	37	139/958	233.8913	223	206.0330
1421	11839	12	44/729	37	139/958	234.0364	223	206.1781
1422	11863	12	7/116	37	139/958	234.1815	223	206.3232
1423	11867	12	7/116	37	139/958	234.3266	224	206.4683
1424	11887	12	7/116	37	139/958	234.4717	224	206.6134
1425	11897	12	7/116	37	139/958	234.6168	224	206.7585
1426	11903	12	7/116	37	139/958	234.7619	224	206.9036
1427	11909	12	7/116	37	139/958	234.9070	225	207.0487
1428	11923	12	54/895	37	139/958	235.0521	225	207.1938
1429	11927	12	54/895	37	139/958	235.1971	226	207.3389
1430	11933	12	33/547	37	139/958	235.3422	226	207.4839
1431	11939	12	40/663	37	139/958	235.4873	226	207.6290
1432	11941	12	26/431	37	139/958	235.6324	226	207.7741
1433	11953	12	26/431	37	139/958	235.7775	227	207.9192
1434	11959	12	45/746	37	139/958	235.9226	227	208.0643
1435	11969	12	45/746	37	139/958	236.0677	227	208.2094
1436	11971	12	19/315	37	139/958	236.2128	227	208.3545
1437	11981	12	19/315	37	139/958	236.3579	227	208.4996
1438	11987	12	31/514	37	139/958	236.5030	227	208.6447
1439	12007	12	31/514	37	139/958	236.6481	228	208.7898
1440	12011	12	12/199	37	139/958	236.7932	228	208.9349
1441	12037	12	12/199	37	139/958	236.9383	228	209.0800
1442	12041	12	12/199	37	139/958	237.0834	228	209.2251
1443	12043	12	12/199	37	139/958	237.2285	228	209.3702
1444	12049	12	53/879	38	139/958	237.3736	228	209.5153
1445	12071	12	53/879	38	139/958	237.5186	228	209.6604
1446	12073	12	29/481	38	139/958	237.6637	228	209.8054
1447	12097	12	29/481	38	139/958	237.8088	229	209.9505
1448	12101	12	17/282	38	139/958	237.9539	229	210.0956
1449	12107	12	17/282	38	139/958	238.0990	229	210.2407
1450	12109	12	56/929	38	139/958	238.2441	229	210.3858
1451	12113	12	17/282	38	139/958	238.3892	230	210.5309
1452	12119	12	22/365	38	139/958	238.5343	230	210.6760
1453	12143	12	22/365	38	139/958	238.6794	231	210.8211
1454	12149	12	49/813	38	139/958	238.8245	231	210.9662
1455	12157	12	49/813	38	139/958	238.9696	231	211.1113
1456	12161	12	59/979	38	139/958	239.1147	231	211.2564
1457	12163	12	59/979	38	139/958	239.2598	231	211.4015
1458	12197	12	37/614	38	139/958	239.4049	231	211.5466
1459	12203	12	37/614	38	139/958	239.5500	232	211.6917
1460	12211	12	47/780	38	139/958	239.6951	232	211.8368
1461	12227	12	47/780	38	139/958	239.8401	232	211.9819
1462	12239	12	5/83	38	139/958	239.9852	232	212.1269
1463	12241	12	5/83	38	139/958	240.1303	232	212.2720
1464	12251	12	5/83	38	139/958	240.2754	232	212.4171
1465	12253	12	5/83	38	139/958	240.4205	232	212.5622
1466	12263	12	5/83	38	139/958	240.5656	232	212.7073
1467	12269	12	5/83	38	139/958	240.7107	232	212.8524
1468	12277	12	5/83	38	139/958	240.8558	232	212.9975
1469	12281	12	5/83	38	139/958	241.0009	232	213.1426
1470	12289	12	5/83	38	139/958	241.1460	232	213.2877
1471	12301	12	5/83	38	139/958	241.2911	233	213.4328
1472	12323	12	48/797	38	139/958	241.4362	233	213.5779
1473	12329	12	53/880	38	139/958	241.5813	233	213.7230
1474	12343	12	38/631	38	139/958	241.7264	233	213.8681

n	p_n	$\pi(\sqrt{n})$	$\prod_{j=1}^{n}(1-1/p_j)$	$[\sqrt{n}]$	$\prod_{j=1}^{\pi(\sqrt{n})}(1-1/p_j)$	$\sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$	$\pi(n)$	$n.\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)$
1475	12347	12	38/631	38	139/958	241.8715	233	214.0132
1476	12373	12	28/465	38	139/958	242.0166	233	214.1583
1477	12377	12	28/465	38	139/958	242.1616	233	214.3034
1478	12379	12	51/847	38	139/958	242.3067	233	214.4484
1479	12391	12	51/847	38	139/958	242.4518	233	214.5935
1480	12401	12	41/681	38	139/958	242.5969	233	214.7386
1481	12409	12	23/382	38	139/958	242.7420	234	214.8837
1482	12413	12	18/299	38	139/958	242.8871	234	215.0288
1483	12421	12	18/299	38	139/958	243.0322	235	215.1739
1484	12433	12	49/814	38	139/958	243.1773	235	215.3190
1485	12437	12	49/814	38	139/958	243.3224	235	215.4641
1486	12451	12	44/731	38	139/958	243.4675	235	215.6092
1487	12457	12	31/515	38	139/958	243.6126	236	215.7543
1488	12473	12	13/216	38	139/958	243.7577	236	215.8994
1489	12479	12	13/216	38	139/958	243.9028	237	216.0445
1490	12487	12	13/216	38	139/958	244.0479	237	216.1896
1491	12491	12	13/216	38	139/958	244.1930	237	216.3347
1492	12497	12	34/565	38	139/958	244.3380	237	216.4798
1493	12503	12	47/781	38	139/958	244.4831	238	216.6248
1494	12511	12	21/349	38	139/958	244.6282	238	216.7699
1495	12517	12	21/349	38	139/958	244.7733	238	216.9150
1496	12527	12	50/831	38	139/958	244.9184	238	217.0601
1497	12539	12	50/831	38	139/958	245.0635	238	217.2052
1498	12541	12	37/615	38	139/958	245.2086	238	217.3503
1499	12547	12	37/615	38	139/958	245.3537	239	217.4954
1500	12553	12	53/881	38	139/958	245.4988	239	217.6405
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Fig.14: The above table compares values for $\pi(n)$ as approximated non-heuristically by $\pi_L(n) = \sum_{j=1}^{n}\prod_{i=1}^{\pi(\sqrt{j})}(1 - 1/p_j)$, the actual values $\pi(n)$ of the primes less than or equal to n, and the values for $\pi(n)$ as estimated non-heuristically by $\pi_H(n) = n.\prod_{i=1}^{\pi(\sqrt{j})}(1 - 1/p_j)$ of $\pi(n)$, for $4 \leq n \leq 1500$.[44]

7.A. Observations and analysis of the error between the actual ($Act\ p$) number $\pi(p_{n+1}^2) - \pi(p_n^2)$ and non-heuristically expected ($Exp\ p$) number $\pi_L(p_{n+1}^2) - \pi_L(p_n^2)$ of primes in the interval (Int) (p_n^2, p_{n+1}^2) for $1 \leq n \leq 11$

Observation and analysis of the error between the actual ($Act\ p$) number, $\pi(p_{n+1}^2) - \pi(p_n^2)$, of primes in the interval (Int) (p_n^2, p_{n+1}^2), and the non-heuristically expected ($Exp\ p$) number, $\pi_L(p_{n+1}^2) - \pi_L(p_n^2)$, of primes in the interval (Int) (p_n^2, p_{n+1}^2) for $1 \leq n \leq 11$ raises the query:

Does the ratio:

$$\mathbb{R} = \frac{CSD}{CExp\ p} = \frac{Cumulative\ standard\ deviation\ of\ the\ cumulative\ sum\ of\ expected\ primes\ in\ the\ interval\ (4,\ p_{n+1}^2)}{Cumulative\ sum\ \pi_L(p_{n+1}^2) = \sum_{j=1}^{p_{n+1}^2}\prod_{i=1}^{\pi(\sqrt{j})}(1-1/p_j)\ of\ expected\ primes\ in\ the\ interval\ (4,\ p_{n+1}^2)}$$

tend to a limit?

Fig.15: Ratio $CSD/CExp\ p = \sum_{i=1}^{n}$ Standard Deviation in $(p_i^2, p_{i+1}^2)/\sum_{i=1}^{n}$ Expected Primes in (p_i^2, p_{i+1}^2)

n	*Interval* $p_n^2 - p_{n+1}^2$	*Int Size*	*Int Act p*	*Int Exp p*	*Int Error*	*Cum Act p*	*Cum Exp p*	*Cum Error*	% *Error*	*Int density* $\prod_1^n(1-\frac{1}{p_i})$	*Int SD*	*Cum SD*	*Ratio* $\mathbb{R}$
1	4 − 9	5	2	1.6	0.4000	2	1.6	0.4000	20.00	0.3333	1.0541	1.0541	0.6588
2	9 − 25	16	5	4.2	0.7714	7	5.8	1.1714	16.73	0.2667	1.7689	2.8230	0.4843
3	25 − 49	24	6	5.5	0.5351	13	11.3	1.7065	13.13	0.2286	2.0571	4.8801	0.4321
4	49 − 121	72	15	14.9	0.0549	28	26.2	1.7614	6.29	0.2078	3.4427	8.3228	0.3172
5	121 − 169	48	9	9.2	−0.1955	37	35.4	1.5659	4.23	0.1918	2.7278	11.0506	0.3119
6	169 − 289	120	22	21.7	0.3465	59	57.1	1.9124	3.24	0.1805	4.2133	15.2640	0.2674
7	289 − 361	72	11	12.3	−1.3063	70	69.4	0.6061	0.87	0.1710	3.1950	18.4589	0.2660
8	361 − 529	168	29	27.5	1.5228	99	96.9	2.1289	2.15	0.1636	4.7945	23.2534	0.2400
9	529 − 841	312	47	49.3	−2.2745	146	146.1	−0.1456	−0.10	0.1579	6.4417	29.6951	0.2032
10	841 − 961	120	16	18.4	−2.3381	162	164.5	−2.4837	−1.53	0.1529	3.9419	33.6371	0.2045
11	961 − 1369	408	57	60.7	−3.6745	219	225.2	−6.1582	−2.81	0.1487	7.1870	40.8241	0.1813
12	1369 − 1500	131	20	19.0	0.9927	239	244.2	−5.1655	−2.16	0.1451	4.0311	44.8551	0.1837
													E & OE

[44] The downloadable .xlsx source file is accessible here.

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References

[Br00] Richard P. Brent. 2000. *Recent Progress and Prospects for Integer Factorisation Algorithms.* In *Computing and Combinatorics*, Lecture Notes in Computer Science, Volume 1858, 2000, pp.3-22, Springer, New York/Heidelberg.

[Cook] Stephen Cook. 2000. *The P versus NP Problem*. Official description provided for the Clay Mathematical Institute, Cambridge, Massachusetts.

[Di52] Leonard Eugene Dickson. 1952. *History of the Theory of Numbers: Volume I.* Chelsea Publishing Company, New York, N. Y.

[Dir37] Peter Gustav Lejeune Dirichlet. 1837. *There are infinitely many prime numbers in all arithmetic progressions with first term and difference coprime.* Originally read to The Royal Prussian Academy of Sciences on the 27th of July, 1837; published in *Abhandlungen der Königlichen Preußischen Akademie der Wissenschaften zu Berlin*, 1837, 4581. *Citation source*: English translation and arXive on 24/11/2014 by Ralf Stephan at http://arXiv:0808.1408v2.

[El79a] P. D.T. A. Elliott. 1979. *Probabilistic Number Theory I.* Springer-Verlag, New York.

[El79b] P. D.T. A. Elliott. 1979. *Probabilistic Number Theory II.* Springer-Verlag, New York.

[Gr95] Andrew Granville. 1995. *Harald Cramér and the distribution of prime numbers*. Scandinavian Actuarial Journal, Volume 1995, Issue 1, pp.12-28. DOI:10.1080/03461238.1995.10413946.

[GS97] Charles M. Grinstead and J. Laurie Snell. 2003. *Introduction to Probability*, *The CHANCE Project* Version dated 4 July 2006 of the Second Revised Edition, 1997, American Mathematical Society, Rhode Island, USA.

[HL23] G.H. Hardy and J.E. Littlewood. 1923. *Some problems of 'partitio numerorum:' III: On the expression of a number as a sum of primes,* Acta Mathematica, December 1923, Volume 44, pp.1-70.

[HW60] G. H. Hardy and E. M. Wright. 1960. *An Introduction to the Theory of Numbers.* 4th edition. Clarendon Press, Oxford.

[Ka59] Mark Kac. 1959. *Statistical Independence in Probability, Analysis and Number Theory.* 1959. *The Carus Mathematical Monographs: Number Twelve* The Mathematical Association of America, Second Impression, 1964.

[Ko56] A. N. Kolmogorov. 1933. *Foundations of the Theory of Probability.* Second English Edition. Translation edited by Nathan Morrison. 1956. Chelsea Publishing Company, New Yourk (*sic*).

[Ri74] Ian Richards. 1974. *On The Incompatibility Of Two Conjectures Concerning Primes; A Discussion Of The Use Of Computers In Attacking A Theoretical Problem.* Bulletin Of The American Mathematical Society, Volume 80, Number 3, May 1974.

[Se49] Atle Selberg. 1949. *An Elementary Proof of Dirichlet's Theorem About Primes in an Arithmetic Progression.* Annals of Mathematics, Second Series, Vol. 50, No. 2 (Apr., 1949), pp. 297-304.

[St02] Jörn Steuding. 2002. *Probabilistic Number Theory.* The Pennsylvania State University CiteSeerX Archives, doi=10.1.1.118.4755.

[Ti51] E. C. Titchmarsh. 1951. *The Theory of the Riemann Zeta-Function.* Clarendon Press, Oxford.

[An14a] Bhupinder Singh Anand. 2014. *A probability-based proof that any deterministic algorithm which always computes a prime factor of n cannot be polynomial-time.* Submitted on 01/01/2015 to the *Calcutta Statistical Association Bulletin*, Kolkata, India.

[An14b] . . . 2014. *Defining prime probability analytically: A probability-based approach to estimating the prime and twin-prime counting functions $\pi(n)$ and $\pi_2(n)$ analytically.* Submitted on 12/12/2014 to the *Calcutta Statistical Association Bulletin*, Kolkata, India.

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