Evidence-Based Interpretations of PA

Peano Arithmetic—Tarskian Interpretability—Computability Theory

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01: Overview

In this paper, we first show that:

- ***In addition*** to the classically defined Standard interpretation of the first order Peano Arithmetic PA . . .
  - ***. . . Tarski's definitions*** also admit two evidence-based interpretations of PA under the standard first order logic FOL:
    - ***An*** Instantiational interpretation; and . . .
    - ***An*** Algorithmic interpretation.

02: Overview (contd.)

We then show that:

- ***The*** Instantiational interpretation of PA is sound if, and only if, the Standard interpretation of PA is sound . . .
    - ***. . . where*** we define an interpretation of PA as sound if, and only if:
        - ***The*** axioms of PA are true under the interpretation; and . . .
        - ***The*** rules of inference preserve such truth.

03: Overview (contd.)

We further show that:

- ***The*** axioms of PA are true under the Algorithmic interpretation; and that . . .
- ***The*** rules of inference preserve such truth.

We conclude that:

- ***The*** Algorithmic interpretation of PA is sound; and that . . .
- ***PA*** is consistent.

04: What is different in our approach?

We approach this investigation by noting that conventional wisdom assumes:

- ***That*** the standard first order logic FOL is consistent; and

  . . .

- ***That*** the standard interpretation of FOL is sound.

05: Conventional wisdom assume Aristotle's particularisation without qualification

- ***The significance*** of this is that conventional wisdom assumes Aristotle's particularisation is valid—without qualification—even over infinite domains.
- ***Aristotle's particularisation*** is the postulation that:
  - ***If*** the formula $[(\exists x)F(x)]$ of a formal language L interprets as true under an interpretation, . . .
  - . . . ***then*** we may always conclude that there must be some term $[a]$ of L such that the formula $[F(a)]$ interprets as true under the interpretation.

06: Constructivists deny both FOL and its standard interpretation

We note that constructive approaches to mathematics—such as Intuitionism—generally do not assume that . . .

- ***FOL*** is consistent.

. . . nor that:

- ***The*** standard interpretation of FOL is sound.

07: Where our approach differs

In this paper, however, we shall assume that ...

- ***FOL*** is consistent.

...but we shall avoid appeal to—and explicitly state when an argument appeals to—the postulation that:

- ***The*** standard interpretation of FOL is sound.

08: A philosophical query

We begin our investigation by asking

- ***Is*** there any objective evidence to justify the acceptance of arithmetical propositions as ‘true’ . . .
    - ***. . . on*** the grounds that such ‘truth’ is self-evident?

09: Conventional wisdom on Tarski and PA

For instance conventional wisdom follows . . .

- ***Tarski's*** inductive definitions, of the 'satisfiability' and 'truth' of the formulas of a formal language under an interpretation;

. . . and postulates that:

- ***The*** Standard interpretation of PA over the domain N of the natural numbers is sound if the standard interpretation of FOL is sound.

10: What is implicitly held as self-evident

What this means is that conventional wisdom:

- ***Holds*** it as self-evident under the standard interpretation of FOL that ...
  - ***...even*** though an infinite process is implicit in the decidability ...
    - ***If*** the formula $[(\exists x)F(x)]$ is true under the Standard interpretation of PA ...
    - ***Then*** there must exist some numeral $[n]$ for which the formula $[F(n)]$ is true under the interpretation.

11: PA and ω-consistency

However, we note for later reference that:

- ***Unless we assume*** that PA is ω-consistent ...
- ***We cannot conclude*** by FOL that ...
    - ***If*** the formula $[(\exists x)F(x)]$ is provable in PA ...
    - ***Then*** there must exist some numeral $[n]$ for which the formula $[F(n)]$ is provable in PA.

12: What is implicitly held further as self-evident

It also means that conventional wisdom:

- ***Holds*** it as self-evident that . . .
  - ***. . . even*** though an infinite process is implicit in their decidability . . .
    - ***The*** denumerable atomic formulas of PA can be assumed as decidable under the Standard interpretation of PA . . .
    - ***The*** denumerable PA axioms can be assumed to interpret as true under the Standard interpretation of PA . . .
    - ***The*** PA rules of inference can be assumed to preserve truth under the Standard interpretation of PA.

13: Algorithmic verifiability

We shall digress for a moment and define the concept of:

Definition

Algorithmic verifiability:

An arithmetical formula $[F(x)]$ is algorithmically verifiable under an interpretation if, and only if, for any given numeral $[n]$, we can define an algorithm AL_n which provides objective evidence for deciding $[F(n)]$ under the interpretation.

- ***Example***: By the definition of a Cauchy sequence . . .
  - ***If*** $[R(n)]$ denotes the n^{th} digit in the decimal expression of the real number R . . .
  - ***Then*** $[R(x)]$ is algorithmically verifiable.

14: Algorithmic computability

We also define the concept of:

Definition

Algorithmic computability:

An arithmetical formula $[F(x)]$ is algorithmically computable under an interpretation if, and only if, we can define an algorithm AL that, for any given numeral $[n]$, provides objective evidence for deciding $[F(n)]$ under the interpretation.

15: A corollary

We note that:

Lemma

Although every algorithmically computable formula is algorithmically verifiable, the converse is not true.

- ***Example***:
  - ***If*** $[R(n)]$ denotes the n^{th} digit in the decimal expression of an uncomputable real number R ...
  - ***Then*** $[R(x)]$ is algorithmically verifiable but not algorithmically computable.

16: Satisfiability and decidability of atomic formulas under three interpretations

Following Tarski's definitions, we can now define the satisfiability of the atomic formulas of PA under three distinctly different interpretations as follows:

17: Satisfiability of atomic formulas under the Standard interpretation of PA

First:

Definition

- ***Under the Standard interpretation***: The natural number n satisfies the atomic formula $[A(x)]$ of PA if, and only if ...
  - ***The*** formula $[A(x)]$ can be assumed to be decidable always as either true or false in N; and ...
  - ***The*** arithmetical statement $A(n)$ is true in N.

18: Satisfiability of atomic formulas under the Instantiational interpretation

Second:

Definition

- ***Under the Instantiational interpretation***: The numeral $[n]$ satisfies the atomic formula $[A(x)]$ of PA if, and only if ...
  - ***The*** formula $[A(x)]$ is algorithmically verifiable as either provable or not provable in PA; and ...
  - ***The*** PA formula $[A(n)]$ is provable in PA.

19: Satisfiability of atomic formulas under the Algorithmic interpretation of PA

Third:

Definition

- ***Under the Algorithmic interpretation***: The natural number n *satisfies* the atomic formula $[A(x)]$ of PA if, and only if ...
  - ***The*** formula $[A(x)]$ is algorithmically computable as either true or false in N; and ...
  - ***The*** arithmetical statement $A(n)$ is true in N.

20: Algorithmic verifiability under the Instantiational interpretation of PA

In this paper we then show that:

Lemma

The atomic formulas of PA are algorithmically verifiable under the Instantiational interpretation.

21: Algorithmic computability under the Algorithmic interpretation of PA

We further show that:

Lemma

The atomic formulas of PA are algorithmically computable under the Algorithmic interpretation.

22: Decidability of compound formulas under Tarski's definitions

- ***Now***, under Tarski's inductive definitions, the compound formulas of PA are decidable under an interpretation if, and only if, the atomic formulas of PA are decidable under the interpretation.

23: The Instantiational and Algorithmic interpretations are valid

We conclude that:

Theorem

The Instantiational and Algorithmic interpretations are valid evidence-based interpretations of PA.

24: Soundness of the Standard interpretation

We recall the conventional wisdom which assumes that:

- ***If*** the formula $[(\exists x)F(x)]$ is true under the Standard interpretation of PA,
- ***Then*** there exists some numeral $[n]$ for which the formula $[F(n)]$ is true under the interpretation; ...

and that:

- ***The*** Standard interpretation of PA is sound.

25: The Instantiational interpretation and ω-consistency

In this paper we first show that:

Lemma

- ***If*** *the PA theorems interpret as true under the Instantiational interpretation of PA . . .*
- ***Then*** *PA is ω-consistent.*

26: The Instantiational interpretation and the Standard interpretation

We then show that:

Lemma

- ***If*** *PA is ω-consistent . . .*
- ***Then*** *a PA formula is true under the Instantiational interpretation . . .*
    - ***If****, and only if, it is true under the Standard interpretation.*

27: Soundness of the Instantiational interpretation

We conclude that:

Theorem

The Instantiational interpretation of PA is sound if, and only if, the Standard interpretation is sound.

28: The Algorithmic interpretation is sound

We also show that:

Lemma

- ***The*** *PA axioms interpret as true arithmetical propositions under the Algorithmic interpretation, and . . .*
- ***The*** *rules of inference preserve such truth.*

29: The Algorithmic interpretation is sound

We conclude that:

Theorem

- ***The** Algorithmic interpretation of PA is sound*
- ***PA** is consistent.*

30: The unquantified axioms of PA are true under the Algorithmic interpretation

We outline the proof of consistency by noting first that:

Lemma

The unquantified axioms of PA are true over N under the Algorithmic interpretation.

Proof: Since $[x + y]$, $[x \star y]$, $[x = y]$, $[x']$ are defined recursively, the unquantified PA axioms interpret as primitive recursive relations. The lemma follows straightforwardly from Tarski's definitions. □

31: The PA axiom schema of finite Induction is true under the Algorithmic interpretation

Moreover, the following PA axiom schema of finite Induction is true under the Algorithmic interpretation:

Lemma

$[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ *is true under the Algorithmic interpretation of PA.*

Proof: By Tarski's definitions:

(a) If $[F(0)]$ is false under the Algorithmic interpretation the lemma is proved.

(b) If $[F(0)]$ is true and $[(\forall x)(F(x) \rightarrow F(x'))]$ is false under the Algorithmic interpretration, the lemma is proved.

32: The PA axiom schema of finite Induction is true under the Algorithmic interpretation

Lemma

$[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ *is true under the Algorithmic interpretation of PA.*

Proof: (continued)

(c) If $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ are true under the Algorithmic interpretation, then there is an algorithm which, for any natural number n, will give evidence that the formula $[F(n) \rightarrow F(n')]$ is true under the interpretation.

Since $[F(0)]$ is true under the Algorithmic interpretation, it follows that there is an algorithm which, for any natural number n, will give evidence that the formula $[F(n)]$ is true under the interpretation.

Hence $[(\forall x)F(x)]$ is true under the Algorithmic interpretation of PA.

Since the above cases are exhaustive, the lemma follows. □

33: Generalisation preserves truth under the Algorithmic interpretation of PA

We further have that:

Lemma

Generalisation preserves algorithmic truth under the Algorithmic interpretation of PA.

Proof: The two meta-assertions:

- '$[F(x)]$ interprets as true under the Algorithmic interpretation of PA' and
- '$[(\forall x)F(x)]$ interprets as true under the Algorithmic interpretation of PA'

both mean:

- $[F(x)]$ is algorithmically computable as always true under the Algorithmic interpretation of PA. □

34: Modus Ponens preserves truth under the Algorithmic interpretation of PA

It also follows immediately from Tarski's definitions that:

Lemma

Modus Ponens preserves truth under the Algorithmic interpretation. □

35: PA is consistent

We thus have that:

Theorem

The axioms of PA are always true under the Algorithmic interpretation of PA, and the rules of inference of PA preserve the properties of satisfaction/truth under the interpretation. □

By definition it follows that:

Theorem

PA is consistent. □

36: Summary

To summarise, we show in this paper:

- ***First***: That Tarski's definitions admit two evidence-based interpretations of PA under FOL:
    - ***An*** Instantiational interpretation of PA; and
    - ***An*** Algorithmic interpretation of PA.
- ***Second***: That the Instantiational interpretation of PA is sound if, and only if, the Standard interpretation of PA is sound.
- ***Third***: That the axioms of PA are true under the Algorithmic interpretation, and the rules of inference preserve such truth.
- ***We conclude that***: The Algorithmic interpretation of PA is sound.
- ***And that***: PA is consistent.

End

That concludes this presentation

Thank you