Is Gödel's undecidable proposition an 'ad hoc' anomaly?

A computational perspective of a general class of Gödelian 'undecidable' arithmetical propositions

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Abstract

We show that if the standard interpretation of PA is sound, then Gödel's arithmetical representation of any recursive relation yields an undecidable PA formula. This Gödelian characteristic is merely a reflection of the fact that, by the instantiational nature of their constructive definition in terms of Gödel's β-function, such formulas are designed to be algorithmically verifiable, but not algorithmically computable, under the standard interpretation of PA.

Keywords Algorithmic computability, algorithmic verifiability, Aristotle's particularisation, arithmetical representation, consistency, first-order, Gödel's β-function, ω-consistency, Peano Arithmetic PA, recursive function, soundness, standard interpretation, undecidable.

1 Introduction

> "... the undecidable proposition 17 *Gen r*, as was remarked at the very beginning, asserts its own unprovability" ... Kurt Gödel, [Go31], p.37, fn.67.

The counter-intuitive element in the widely accepted conclusion[1] of self-referential causality that Kurt Gödel has drawn from Theorem VI[2] of his seminal 1931 paper—on formally undecidable arithmetical propositions—has occasionally given rise to the perception that Gödel's 'undecidable' proposition $[(\forall x)R(x)]$[3] is an

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[1] This is the conclusion that Gödel's 'undecidable' arithmetical proposition—which is of the form $[(\forall x)R(x)]$—is undecidable because it involves self-reference; in the sense that, under a sound interpretation of the arithmetic, it can be taken to assert that 'The arithmetical formula with Gödel number $\lceil(\forall x)R(x)\rceil$ is not provable in the arithmetic', where $\lceil F\rceil$ denotes the Gödel number of the arithmetical formula $[F]$.

[2] "For every ω-consistent recursive class κ of FORMULAS, there exists a recursive CLASS EXPRESSION r such that neither v *Gen* r nor $Neg(v$ *Gen* $r)$ belongs to $Flg(\kappa)$ (where v is the FREE VARIABLE of r". [Go31], p.24, Theorem VI.

[3] We note that $[R(x)]$ is the standard arithmetical representation of a standard primitive recursive—seemingly 'circularly' defined—relation $Q(x)$. In his seminal 1931 paper [Go31], Kurt Gödel defines, and refers to, the formula corresponding to $[R(x)]$ only by its 'Gödel' number r (op. cit., p.25, Eqn.(12)), and to the formula corresponding to $[(\forall x)R(x)]$ only by its 'Gödel' number 17 *Gen* r (op. cit., p.25, Eqn.(13)).

artificially constructed 'ad hoc'[4] anomaly, the ilk of which is not likely to be encountered in, or have any appreciable significance for, the classical perspective of mainstream mathematics.

We shall argue that, to some extent, such a perception of self-reference (and perhaps even of diagonalisation) as the—or even as a necessary—cause of undecidability may be misleading.

For instance, in the Gödelian case $[R(x)]$ is the arithmetical representation of a primitive recursive relation and we shall show that, if we assume the standard interpretation of PA is sound (in the sense of Definition 10), then Gödel's arithmetical representation of any recursive relation yields a similarly undecidable PA formula.

Specifically we shall show that, from a computational perspective, this Gödelian characteristic is merely a reflection of the fact that, by the instantiational nature of their constructive definition in terms of Gödel's β-function, such formulas are designed to be algorithmically verifiable (Definition 11), but not algorithmically computable (Definition 12), under the standard interpretation of PA.[5]

Expressed formally, it follows from Gödel's Theorem VII[6] that the first part[7] of his preceding Theorem VI is merely a special case of the following lemma, which is a consequence of the central argument—Lemma 9 and Corollary 1—of this investigation:

Lemma 1 *If PA is consistent and* $[F(x_1, x_2, x_3)]$ *is the arithmetical formula*[8] *that represents the recursive function* $f(x_1, x_2)$ *in PA, then:*

> *(a)* $[(\exists_1 x_3)F(k, m, x_3)]$[9] *is PA-provable for any given numerals* $[k], [m]$*;*
>
> *(b)* $[(\exists_1 x_3)F(x_1, x_2, x_3)]$ *is not PA-provable.*[10]
>
> **Notation** We use square brackets to indicate that the contents represent a symbol or a formula—of a formal theory—generally assumed to be well-formed unless otherwise indicated by the context.[11] We use an asterisk to indicate that the associated expression is to be interpreted semantically with respect to some well-defined interpretation.

Proof (a) By definition[12] $[(\exists_1 x_3)F(k, m, x_3)]$ is PA-provable for any given numerals $[k], [m]$ since the PA formula $[F(x_1, x_2, x_3)]$ represents the recursive function $f(x_1, x_2)$.

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[4] See for instance [Lg11]

[5] From this perspective, the significance of Gödel's argumentation in [Go31] is that formalisation of the distinction between 'algorithmic verifiability' and 'algorithmic computability' leads to two distinctly different—and hitherto unsuspected—finitary interpretations of the first order Peano Arithmetic PA which, in turn, lead to the finitary proof of consistency for PA detailed in [An13].

[6] "Every relation of the form $x_0 = \phi(x_1, \ldots, x_n)$, where ϕ is recursive, is arithmetical …". [Go31], p.29, Theorem VII; see Section 7 for a detailed excerpt.

[7] "17 *Genr* is not κ-PROVABLE", [Go31], p.25(1).

[8] As defined in Lemma 7.

[9] Where '$\exists_1$' denotes uniqueness.

[10] This appears to settle the following 'open' question ([Me64], p.135, Ex.3) negatively: Is every recursive function strongly representable in PA?

[11] In other words, expressions inside the square brackets are to be only viewed syntactically as juxtaposition of symbols that are to be formed and manipulated upon strictly in accordance with specific rules for such formation and manipulation—in the manner of a mechanical or electronic device—without any regards to what the symbolism might represent semantically under an interpretation that gives them meaning.

[12] Definition 13.

(b) If $[(\exists_1 x_3)F(x_1, x_2, x_3)]$ were PA-provable, then it would follow from the preliminary Lemma 4 below that $[(\exists_1 x_3)F(x_1, x_2, x_3)]$ is also algorithmically computable as true under the standard interpretation of PA (Definition 6). We shall show in Lemma 9 that this is not the case. The lemma follows. □

If we further assume that the standard interpretation of PA is sound[13], then it follows from para (a) in the preceding proof that the second part[14] of Gödel's Theorem VI is a special case of the following:

Lemma 2 *If the standard interpretation of PA is sound and* $[F(x_1, x_2, x_3)]$ *is the arithmetical formula that represents the recursive function* $f(x_1, x_2)$ *in PA, then* $[\neg(\exists_1 x_3)F(x_1, x_2, x_3)]$ *is not PA-provable.*

Proof $[\neg(\exists_1 x_3)F(x_1, x_2, x_3)]$ is an abbreviation of the PA formula:

$$[(\forall x_3)\neg F(x_1, x_2, x_3) \wedge (\forall y)(\forall z)(F(x_1, x_2, y) \wedge F(x_1, x_2, z) \rightarrow y = z)].$$

Under any sound interpretation of PA over $\mathbb{N}$, the latter formula interprets as the arithmetical relation denoted by:

$$(\forall x_3)\neg F^*(x_1, x_2, x_3) \wedge (\forall y)(\forall z)(F^*(x_1, x_2, y) \wedge F^*(x_1, x_2, z) \rightarrow y = z).$$

Now, if the standard interpretation of PA is sound then Aristotle's particularisation (Definition 1) holds over $\mathbb{N}$[15], and this relation can be equivalently denoted by:

$$\neg(\exists x_3)F^*(x_1, x_2, x_3) \wedge (\forall y)(\forall z)(F^*(x_1, x_2, y) \wedge F^*(x_1, x_2, z) \rightarrow y = z).$$

It follows that if $[\neg(\exists_1 x_3)F(x_1, x_2, x_3)]$ is PA-provable, then $\neg(\exists x_3)F^*(x_1, x_2, x_3)$ is always true over $\mathbb{N}$.

However, this is false since $(\exists x_3)F^*(x_1, x_2, x_3)$ is always true over $\mathbb{N}$ by definition[16]. The lemma follows. □

We conclude that the proof of Gödel's Theorem VI[17] is a special case of:

Theorem 1 *If the standard interpretation of PA is sound and* $[F(x_1, x_2, x_3)]$ *is the arithmetical formula that represents the recursive function* $f(x_1, x_2)$ *in PA, then* $[(\exists_1 x_3)F(x_1, x_2, x_3)]$ *is undecidable in PA.*

Proof The theorem follows from Lemma 1(b) above and Lemma 2. □

2 Comments, Notation and Definitions

Comments We have taken some liberty in emphasising standard definitions selectively, and interspersing our arguments liberally with comments and references, generally of a foundational nature. These are intended to reflect our

[13] This assumption corresponds to Gödel's assumption of ω-consistency. Its significance lies in the fact that the standard interpretation of PA is sound if, and only if, PA is ω-consistent (see [An12]). Although not obvious, such an assumption is subsumed in J. Barkley Rosser's proof of undecidability ([Ro36]), which implicitly assumes (see [An09] Aristotle's particularisation (Definition 1) when addressing the existential quantifier in Rosser's 'undecidable' proposition.

[14] "*Neg*(17 *Genr*) is not κ-PROVABLE", [Go31], p.25(2).

[15] The proof is not obvious, and is detailed in[An12].

[16] Definition 13.

[17] [Go31], p. 24.

underlying thesis that essentially arithmetical problems appear more natural when expressed—and viewed—within the perspective[18] of an interpretation of PA that appeals to the *evidence* provided by a deterministic algorithm[19]; a perspective that, by its very nature, cannot appeal implicitly to transfinite concepts.

> **Evidence** "It is by now folklore ... that one can view the *values* of a simple functional language as specifying *evidence* for propositions in a constructive logic ..."[20].

Definition 1 ***Aristotle's particularisation*** *This holds that from an assertion such as:*

'It is not the case that: For any given x, $P^*(x)$ does not hold',

usually denoted symbolically by '$\neg(\forall x)\neg P^*(x)$', we may always validly infer in the classical, Aristotlean, logic of predicates[21] *that:*

'There exists an unspecified x such that $P^*(x)$ holds',

usually denoted symbolically by '$(\exists x)P^*(x)$'.

The significance of Aristotle's particularisation for the first-order predicate calculus: We note that in a formal language the formula '$[(\exists x)P(x)]$' is an abbreviation for the formula '$[\neg(\forall x)\neg P(x)]$'. The commonly accepted interpretation of this formula—and a fundamental tenet of classical logic unrestrictedly adopted as intuitively obvious by standard literature[22] that seeks to build upon the formal first-order predicate calculus—tacitly appeals to Aristotlean particularisation.

However, L. E. J. Brouwer had noted in his seminal 1908 paper on the unreliability of logical principles[23] that the commonly accepted interpretation of this formula is ambiguous if interpretation is intended over an infinite domain.

Brouwer essentially argued that, even supposing the formula '$[P(x)]$' of a formal Arithmetical language interprets as an arithmetical relation denoted by '$P^*(x)$', and the formula '$[\neg(\forall x)\neg P(x)]$' as the arithmetical proposition denoted by '$\neg(\forall x)\neg P^*(x)$', the formula '$[(\exists x)P(x)]$' need not interpret as the arithmetical proposition denoted by the usual abbreviation '$(\exists x)P^*(x)$'; and that such postulation is invalid as a general logical principle in the absence of a means for constructing some putative object a for which the proposition $P^*(a)$ holds in the domain of the interpretation.

Hence we shall follow the convention that the assumption that '$(\exists x)P^*(x)$' is the intended interpretation of the formula '$[(\exists x)P(x)]$'—which is essentially the assumption that Aristotle's particularisation holds over the domain of the interpretation—must always be explicit.

The significance of Aristotle's particularisation for PA: In order to avoid intuitionistic objections to his reasoning, Kurt Gödel introduced the syntactic

[18] Detailed in [An12].

[19] A deterministic algorithm has only one possible move from a given configuration.

[20] [Mu91].

[21] [HA28], pp.58-59.

[22] See [Hi25], p.382; [HA28], p.48; [Sk28], p.515; [Go31], p.32.; [Kl52], p.169; [Ro53], p.90; [BF58], p.46; [Be59], pp.178 & 218; [Su60], p.3; [Wa63], p.314-315; [Qu63], pp.12-13; [Kn63], p.60; [Co66], p.4; [Me64], p.52(ii); [Nv64], p.92; [Li64], p.33; [Sh67], p.13; [Da82], p.xxv; [Rg87], p.xvii; [EC89], p.174; [Mu91]; [Sm92], p.18, Ex.3; [BBJ03], p.102.

[23] [Br08].

property of ω-consistency[24] as an explicit assumption in his formal reasoning in his seminal 1931 paper on formally undecidable arithmetical propositions[25].

Gödel explained at some length[26] that his reasons for introducing ω-consistency explicitly was to avoid appealing to the semantic concept of classical arithmetical truth in Aristotle's logic of predicates (which presumes Aristotle's particularisation).

The two concepts are meta-mathematically equivalent in the sense that, if PA is consistent, then PA is ω-consistent if, and only if, Aristotle's particularisation holds under the standard interpretation of PA[27].

Definition 2 ***The structure*** $\mathbb{N}$ *The structure of the natural numbers—namely,* $\{N$ *(the set of natural numbers);* $=$ *(equality);* $'$ *(the successor function);* $+$ *(the addition function);* $*$ *(the product function);* 0 *(the null element)*$\}$.

Definition 3 ***The axioms of first-order Peano Arithmetic (PA)***

$\mathbf{PA}_1$ $[(x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))]$;
$\mathbf{PA}_2$ $[(x_1 = x_2) \rightarrow (x_1' = x_2')]$;
$\mathbf{PA}_3$ $[0 \neq x_1']$;
$\mathbf{PA}_4$ $[(x_1' = x_2') \rightarrow (x_1 = x_2)]$;
$\mathbf{PA}_5$ $[(x_1 + 0) = x_1]$;
$\mathbf{PA}_6$ $[(x_1 + x_2') = (x_1 + x_2)']$;
$\mathbf{PA}_7$ $[(x_1 \star 0) = 0]$;
$\mathbf{PA}_8$ $[(x_1 \star x_2') = ((x_1 \star x_2) + x_1)]$;
$\mathbf{PA}_9$ For any well-formed formula $[F(x)]$ of PA:
$[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$.

Definition 4 ***Generalisation in PA*** *If* $[A]$ *is PA-provable, then so is* $[(\forall x)A]$.

Definition 5 ***Modus Ponens in PA*** *If* $[A]$ *and* $[A \rightarrow B]$ *are PA-provable, then so is* $[B]$.

Definition 6 ***Standard interpretation of PA*** *The standard interpretation* $\mathcal{I}_{PA(\mathbb{N},\ Standard)}$ *of PA over the structure* $\mathbb{N}$ *is the one in which the logical constants have their 'usual' interpretations*[28] *in Aristotle's logic of predicates (which subsumes Aristotle's particularisation), and*[29]*:*

(a) the set of non-negative integers is the domain;
(b) the symbol [0] interprets as the integer 0;
(c) the symbol $[']$ *interprets as the successor operation (addition of 1);*
(d) the symbols $[+]$ *and* $[*]$ *interpret as ordinary addition and multiplication;*
(e) the symbol $[=]$ *interprets as the identity relation.*

Definition 7 ***Simple consistency:*** *A formal system S is simply consistent if, and only if, there is no S-formula* $[F(x)]$ *for which both* $[(\forall x)F(x)]$ *and* $[\neg(\forall x)F(x)]$ *are S-provable.*

[24] The significance of ω-consistency for the formal system PA is highlighted in[An12].
[25] [Go31], p.23 and p.28.
[26] In his introduction on p.9 of [Go31].
[27] For details see [An12].
[28] See [Me64], p.49.
[29] See [Me64], p.107.

Definition 8 ***ω-consistency:*** *A formal system S is ω-consistent if, and only if, there is no S-formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S-provable and, second, $[F(a)]$ is S-provable for any given S-term $[a]$.*

Definition 9 ***Soundness (formal system - non-standard):*** *A formal system S is sound under an interpretation $\mathcal{I}_S$ with respect to a domain $\mathbb{D}$ if, and only if, every theorem $[T]$ of S translates as '$[T]$ is true under $\mathcal{I}_S$ in $\mathbb{D}$'.*

Definition 10 ***Soundness (interpretation - non-standard):*** *An interpretation $\mathcal{I}_S$ of a formal system S is sound with respect to a domain $\mathbb{D}$ if, and only if, S is sound under the interpretation $\mathcal{I}_S$ over the domain $\mathbb{D}$.*

Soundness in classical logic: In classical logic, a formal system S is sometimes defined as 'sound' if, and only if, it has an interpretation; and an interpretation is defined as the assignment of meanings to the symbols, and truth-values to the sentences, of the formal system. Moreover, any such interpretation is defined as a model[30] of the formal system. This definition suffers, however, from an implicit circularity: the formal logic L underlying any interpretation of S is implicitly assumed to be 'sound'. The above definitions seek to avoid this implicit circularity by delinking the defined 'soundness' of a formal system under an interpretation from the implicit 'soundness' of the formal logic underlying the interpretation. This admits the case where, even if L_1 and L_2 are implicitly assumed to be sound, $S + L_1$ is sound, but $S + L_2$ is not. Moreover, an interpretation of S is now a model for S if, and only if, it is sound.[31]

Definition 11 *A number-theoretical relation $F(x)$ is algorithmically verifiable if, and only if, for any given natural number n, there is an algorithm $AL_{(F,\ n)}$ which can provide objective evidence[32] for deciding the truth/falsity of each proposition in the finite sequence $\{F(1), F(2), \ldots, F(n)\}$.*

Tarskian interpretation of an arithmetical language *verifiably* in terms of the computations of a simple functional language We show in [An12] that the 'algorithmic verifiability' of the formulas of a formal language which contain logical constants can be inductively defined under an interpretation in terms of the 'algorithmic verifiability' of the interpretations of the atomic formulas of the language; further, that the PA-formulas are decidable under the standard interpretation of PA over N if, and only if, they are algorithmically verifiable under the interpretation.[33]

Definition 12 *A number theoretical relation $F(x)$ is algorithmically computable if, and only if, there is an algorithm AL_F that can provide objective evidence for deciding the truth/falsity of each proposition in the denumerable sequence $\{F(1), F(2), \ldots\}$.*

Tarskian interpretation of an arithmetical language *algorithmically* in terms of the computations of a simple functional language We show in [An12] that the 'algorithmic computability' of the formulas of a formal language which contain logical constants can also be inductively defined under an interpretation in terms of the 'algorithmic computability' of the interpretations of the atomic formulas of the language; further, that the PA-formulas are decidable under an algorithmic interpretation of PA over N if, and only if, they are algorithmically computable under the interpretation.[34]

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[30] We follow the definition in [Me64], p.51.

[31] My thanks to Professor Rohit Parikh for highlighting the need for making such a distinction explicit.

[32] cf. [Mu91]: "It is by now folklore ... that one can view the *values* of a simple functional language as specifying *evidence* for propositions in a constructive logic ...".

[33] We show in [An12] that the concept of Algorithmic verifiability is also well-defined under the standard interpretation of PA over N.

[34] We show in [An12] that the concepts of Algorithmic verifiability and Algorithmic computability are both well-defined under the standard interpretation of PA over N; moreover they identify distinctly different subsets of the well-defined PA formulas.

Algorithmic verifiability vis à vis algorithmic computability We note that algorithmic computability implies the existence of an algorithm that can decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions[35], whereas algorithmic verifiability does not imply the existence of an algorithm that can decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions.

From the point of view of a finitary mathematical philosophy—which is the constraint within which an applied science ought to ideally operate—the significant difference between the two concepts could be expressed by saying that we may treat the decimal representation of a real number as corresponding to a physically measurable limit[36]—and not only to a mathematically definable limit—if and only if such representation is definable by an algorithmically computable function.[37]

We note that although every algorithmically computable relation is algorithmically verifiable, the converse is not true.[38]

3 Two preliminary lemmas

We begin our investigation by noting two preliminary lemmas:

Lemma 3 *If PA has a sound interpretation $\mathcal{I}_{PA(\mathbb{N},\ Sound)}$ over $\mathbb{N}$, then any PA-provable formula $[F(x_1, \ldots, x_n)]$ is algorithmically verifiable as always true over N under $\mathcal{I}_{PA(\mathbb{N},\ Sound)}$.*

Proof Gödel has shown how we can algorithmically assign a unique natural (Gödel) number to each PA formula and to each finite sequence of PA formulas[39]. Gödel has also shown how we can construct a primitive recursive relation xBy[40] that holds if, and only if, x is the Gödel number of a proof sequence in PA, and y is the Gödel number of the last formula of the sequence.

Now, if the PA formula $[F(x_1, \ldots, x_n)]$ is PA-provable then, for any given sequence of numerals $[(a_1, \ldots, a_n)]$, the PA formula $[F(a_1, \ldots, a_n)]$ is PA-provable. Hence $xB\ulcorner[F(a_1, \ldots, a_n)]\urcorner$[41] always holds for some x. Since xBy is recursive, we can define an algorithm AM_B that will accept m if m is $\ulcorner[F(a_1, \ldots, a_n)]\urcorner$ and certify $[F(a_1, \ldots, a_n)]$ as PA provable.

Since a PA-provable formula is true under $\mathcal{I}_{PA(\mathbb{N},\ Sound)}$, the lemma follows. □

Lemma 4 *If PA has a sound interpretation $\mathcal{I}_{PA(\mathbb{N},\ Sound)}$ over $\mathbb{N}$, then any PA-provable formula $[F(x_1, \ldots, x_n)]$ is algorithmically computable as always true over N.*

Proof[42] If a PA formula $[F(x_1, \ldots, x_n)]$ is PA-provable, then there is a finite proof sequence in PA whose last member is $[F(x_1, \ldots, x_n)]$. Under any sound interpretation of PA over $\mathbb{N}$ this sequence must interpret as an algorithm (program) of fixed size that, for any sequence $[(a_1, \ldots, a_n)]$ of PA numerals, certifies $[F(a_1, \ldots, a_n)]$ as true.

The proposition follows. □

[35]We note that the concept of 'algorithmic computability' is essentially an expression of the more rigorously defined concept of 'realizability' in [Kl52], p.503.

[36]In the sense of a physically 'completable' infinite sequence (as needed to resolve Zeno's paradox).

[37]This point is addressed in more detail in [An13].

[38]See Appendix B, Section §7.

[39][Go31], p.13.

[40][Go31], p.22(45)

[41]$\ulcorner[F(a_1, \ldots, a_n)]\urcorner$ denotes the Gödel number of $[F(a_1, \ldots, a_n)]$.

[42]For a more formal proof see [An12].

4 Gödel's Theorems VI and formally unprovable but interpretively true propositions

We begin by noting that in Theorem V of his 1931 paper[43] Gödel informally proved that every recursive relation $f(x_1, \ldots, x_n)$ can be expressed in PA constructively by a formula $[F(x_1, \ldots, x_n)]$ such that, for any given n-tuple of natural numbers $a_1, \ldots, a_n$:

If $f(a_1, \ldots, a_n)$ is true, then PA proves $[F(a_1, \ldots, a_n)]$

If $\neg f(a_1, \ldots, a_n)$ is true, then PA proves $[\neg F(a_1, \ldots, a_n)]$

Gödel relied only on the above to conclude—in his Theorem VI[44]—the existence of an arithmetical proposition that is formally unprovable in a Peano Arithmetic, but true under any sound interpretation of the Arithmetic.

4.1 Gödel's Theorem VII and the standard interpretation of PA

However, we now show that it is Gödel's Theorem VII[45] which—for recursive relations of the form $x_0 = \phi(x_1, \ldots, x_n)$ defined by the Recursion Rule[46]—provides an actual blueprint for the construction of PA formulas that are PA-unprovable, but true under the standard interpretation of PA.

Moreover, we shall show that this Gödelian characteristic is merely a reflection of the fact that, by the instantiational nature of their constructive definition in terms of Gödel's β-function, such formulas are designed to be algorithmically verifiable, but not algorithmically computable, under the standard interpretation of PA.

(We note that some standard definitions and results in what follows implicitly presume[47] that the standard interpretation of PA is sound, hence quantifiers are interpreted under the assumption that Aristotle's particularisation is valid over $\mathbb{N}$.)

Now, in Theorem VII of his 1931 paper Gödel defined[48] a primitive recursive function—Gödel's β-function—as:

$$\beta(x_1, x_2, x_3) = rm(1 + (x_3 + 1) \star x_2, x_1)$$

where $rm(x_1, x_2)$ denotes the remainder obtained on dividing x_2 by x_1.

Gödel then showed that:

Lemma 5 *For any non-terminating sequence of values $f(x_1, 0), f(x_1, 1), \ldots$, we can construct natural numbers b, c such that:*

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[43][Go31], p.22.

[44][Go31], p.24.

[45][Go31], p.29.

[46][Me64], p.120 & p.132.

[47]Such an implicit presumption is seen in Gödel's reference in the statement of his Theorem IX to the negation of a universally quantified formula of the restricted functional calculus as indicative of "the existence of a counter-example" ([Go31], p.32).

[48]cf. [Go31], p.31, Lemma 1; [Me64], p.131, Proposition 3.21.

(i) $j = max(n, f(x_1, 0), f(x_1, 1), \ldots, f(x_1, n))$;

(ii) $c = j!$;

(iii) $\beta(b, c, i) = f(x_1, i)$ *for* $0 \leq i \leq n$.

Proof This is a standard result[49]. We reproduce Gödel's original argument of this critical lemma in an Appendix A, Section 7. □

Now we have the standard definition[50]:

Definition 13 *A number-theoretic function* $f(x_1, \ldots, x_n)$ *is said to be representable in PA if, and only if, there is a PA formula* $[F(x_1, \ldots, x_{n+1})]$ *with the free variables* $[x_1, \ldots, x_{n+1}]$*, such that, for any given natural numbers* $k_1, \ldots, k_{n+1}$*:*

(i) if $f(k_1, \ldots, k_n) = k_{n+1}$ *then PA proves:* $[F(k_1, \ldots, k_n, k_{n+1})]$*;*

(ii) PA proves: $[(\exists_1 x_{n+1})F(k_1, \ldots, k_n, x_{n+1})]$.

The function $f(x_1, \ldots, x_n)$ *is said to be strongly representable in PA if we further have that:*

(iii) PA proves: $[(\exists_1 x_{n+1})F(x_1, \ldots, x_n, x_{n+1})]$

We then have:

Lemma 6 $\beta(x_1, x_2, x_3)$ *is strongly represented in PA by* $[Bt(x_1, x_2, x_3, x_4)]$*, which is defined as follows:*

$$[(\exists w)(x_1 = ((1 + (x_3 + 1) \star x_2) \star w + x_4) \wedge (x_4 < 1 + (x_3 + 1) \star x_2))].$$

Proof This is a standard result[51]. □

Gödel further showed that:

Lemma 7 *If* $f(x_1, x_2)$ *is a recursive function defined by:*

(i) $f(x_1, 0) = g(x_1)$

(ii) $f(x_1, (x_2 + 1)) = h(x_1, x_2, f(x_1, x_2))$

where $g(x_1)$ *and* $h(x_1, x_2, x_3)$ *are recursive functions of lower rank*[52] *that are represented in PA by well-formed formulas* $[G(x_1, x_2)]$ *and* $[H(x_1, x_2, x_3, x_4)]$*, then* $f(x_1, x_2)$ *is represented in PA by the following well-formed formula, denoted by* $[F(x_1, x_2, x_3)]$*:*

$$[(\exists u)(\exists v)(((\exists w)(Bt(u, v, 0, w) \wedge G(x_1, w))) \wedge Bt(u, v, x_2, x_3) \wedge (\forall w)(w < x_2 \rightarrow (\exists y)(\exists z)(Bt(u, v, w, y) \wedge Bt(u, v, (w + 1), z) \wedge H(x_1, w, y, z)))].$$

Proof This is a standard result[53]. In view of the significance of this lemma for Lemma 9 below, we reproduce Gödel's original argument and proof of the lemma in Appendix A, Section 7. □

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[49] cf. [Go31], p.31, p.31, Lemma 1; [Me64], p.131, Proposition 3.22.

[50] [Me64], p.118.

[51] cf. [Me64], p.131, proposition 3.21.

[52] cf. [Me64], p.132; [Go31], p.30(2).

[53] cf. [Go31], p.31(2); [Me64], p.132.

4.2 What does "$[(\exists_1 x_3)F(k,m,x_3)]$ is provable" assert under the standard interpretation of PA?

Now, if the PA formula $[F(x_1,x_2,x_3)]$ represents in PA the recursive function denoted by $f(x_1,x_2)$ then by definition, for any given numerals $[k,m]$, the formula $[(\exists_1 x_3)F(k,m,x_3)]$ is provable in PA; and true under any sound interpretation of PA. We thus have that:

Lemma 8 *"$[(\exists_1 x_3)F(k,\ m,x_3)]$ is true under the standard interpretation of PA" is the assertion that:*

> *Given any natural numbers k,m, we can construct natural numbers $t_{(k,m)}$, $u_{(k,m)}$, $v_{(k,m)}$—all functions of k,m—such that:*
>
> *(a) $\beta(u_{(k,m)},v_{(k,m)},0)=g(k)$;*
>
> *(b) for all $i<m$, $\beta(u_{(k,m)},v_{(k,m)},i)=h(k,i,f(k,i))$;*
>
> *(c) $\beta(u_{(k,m)},v_{(k,m)},m)=t_{(k,m)}$;*
>
> *where $f(x_1,x_2)$, $g(x_1)$ and $h(x_1,x_2,x_3)$ are any recursive functions that are formally represented in PA by $F(x_1,x_2,\ x_3),G(x_1,x_2)$ and $H(x_1,x_2,x_3,\ x_4)$ respectively such that:*
>
> *(i) $f(k,0)=g(k)$*
>
> *(ii) $f(k,(y+1))=h(k,y,f(k,y))$ for all $y<m$*
>
> *(iii) $g(x_1)$ and $h(x_1,x_2,x_3)$ are recursive functions that are assumed to be of lower rank than $f(x_1,x_2)$.*

Proof For any given natural numbers k and m, if $[F(x_1,x_2,x_3)]$ interprets as a well-defined arithmetical relation under the standard interpretation of PA, then we can define an algorithm that will construct the sequences $f(k,0),f(k,1),\ldots,$ $f(k,m)$ and $\beta(u_{(k,m)},v_{(k,m)},0),\beta(u_{(k,m)},v_{(k,m)},1),\ldots,\beta(u_{(k,m)},v_{(k,m)},m)$ and verify the assertion. □

We now see that:

Lemma 9 *If the standard interpretation of PA is sound, then $[(\exists_1 x_3)F(x_1,x_2,$ $x_3)]$ is algorithmically verifiable, but not algorithmically computable, as always true over $\mathbb{N}$.*

Proof We assume that the standard interpretation of PA is sound (hence we may, for instance, conclude 'There is some x such that ...' from the assertion 'It is not the case that: for all x it is not the case that ...' in the domain $\mathbb{N}$ of the interpretation.). It then follows from Lemma 8 that:

> (1) $[(\exists_1 x_3)F(k,m,x_3)]$ is PA-provable for any given numerals $[k,m]$. Hence $[(\exists_1 x_3)F(k,m,x_3)]$ is true under the standard interpretation of PA. It then follows from the definition of $[F(x_1,x_2,x_3)]$ in Lemma 7 that, for any given natural numbers k,m, we can construct some pair of natural numbers $u_{(k,m)},v_{(k,m)}$—where $u_{(k,m)},v_{(k,m)}$ are functions of the given natural numbers k and m—such that:

(a) $\beta(u_{(k,m)}, v_{(k,m)}, i) = f(k, i)$ for $0 \leq i \leq m$;

(b) $F^*(k, m, f(k, m))$ holds in $\mathbb{N}$.

Since $\beta(x_1, x_2, x_3)$ is primitive recursive, $\beta(u_{(k,m)}, v_{(k,m)}, i)$ defines a constructible non-terminating sequence $f_{(k,m)}(k, 0), f_{(k,m)}(k, 1), \ldots$ for any given natural numbers k and m such that:

(c) $f(k, i) = f_{(k,m)}(k, i)$ for $0 \leq i \leq m$.

We can thus define an algorithm $TM_{\beta(u_{(k,m)}, v_{(k,m)}, i)}$ that will accept the natural number input g if g is the Gödel number of the PA formula $[(\exists_1 x_3)F(k, m, x_3)]$, and then halt with output 'true'.

Hence $[(\exists_1 x_3)F(x_1, x_2, x_3)]$ is algorithmically verifiable as always true over $\mathbb{N}$ under the standard interpretation of PA.

(2) Now, the pair of natural numbers $u_{(x_1,x_2)}, v_{(x_1,x_2)}$ are defined such that:

(a) $\beta(u_{(x_1,x_2)}, v_{(x_1,x_2)}, i) = f(x_1, i)$ for $0 \leq i \leq x_2$;

(b) $F^*(x_1, x_2, f(x_1, x_2))$ holds in $\mathbb{N}$;

where $v_{(x_1,x_2)}$ is defined in Lemma 7 as $j!$ (see Lemma 5), and:

(c) $j = max(n, f(x_1, 0), f(x_1, 1), \ldots, f(x_1, x_2))$;

(d) n is the 'number' of terms in the sequence $f(x_1, 0), f(x_1, 1), \ldots, f(x_1, x_2)$.

Since j is not definable for a non-terminating sequence $\beta(u_{(k,x_2)}, v_{(k,x_2)}, i)$ we cannot construct a non-terminating sequence $f_{(k,x_2)}(k, 0), f_{(k,x_2)}(k, 1), \ldots$ such that:

(e) $f(k, i) = f_{(k,x_2)}(k, i)$ for all $i \geq 0$.

We cannot thus define an algorithm $TM_{\beta(u_{(k,x_2)}, v_{(k,x2)}, i)}$ that will accept the natural number input g if, and only if, g is the Gödel number of the PA formula $[(\exists_1 x_3)F(k, m, x_3)]$, and then halt with output 'true'.

Hence $[(\exists_1 x_3)F(x_1, x_2, x_3)]$ is not computable algorithmically as always true over $\mathbb{N}$ under the standard interpretation of PA.

The lemma follows. □

A critical issue that we do not address in this investigation is whether the PA formula $[F(x_1, x_2, x_3]$ can be considered to interpret under a sound interpretation of PA as a well-defined predicate since the denumerable sequences $f'(k, 0), f'(k, 1), \ldots, f'(k, m), m_p$—where $p > 0$, and m_p is not equal to m_q if p is not equal to q—are represented by denumerable, distinctly different, functions $\beta(x_{p_1}, x_{p_2}, i)$ respectively. There are thus denumerable pairs (x_{p_1}, x_{p_2}) for which $\beta(x_{p_1}, x_{p_2}, i)$ yields any given sequence $f'(k, 0), f'(k, 1), \ldots, f'(k, m)$.

It also follows from the preceding section that:

Corollary 1 *If the standard interpretation of PA is sound, then* $[(\exists_1 x_3)F(x_1, x_2, x_3)]$ *is not PA-provable.*

Proof If $[(\exists_1 x_3)F(x_1, x_2, x_3)]$ were PA-provable then, by Lemma 4, there would be an algorithm that decides $[(\exists_1 x_3)F(x_1, x_2, x_3)]$ as always true under the standard interpretation of PA. By Lemma 9 this is not the case. The corollary follows. □

5 Conclusion

We conclude that Gödel's Theorem VI on the existence of an 'undecidable' arithmetical proposition is a special case of the more general result that, if the standard interpretation of PA is sound, then Gödel's arithmetical representation of any recursive relation yields an undecidable PA formula.

This Gödelian characteristic is a reflection of the fact that, by the instantiational nature of their constructive definition in terms of Gödel's β-function, such formulas are designed to be algorithmically verifiable, but not algorithmically computable, under the standard interpretation of PA.

6 Appendix A: Gödel's Theorem VII(2)

(Excerpted from [Go31] pp.29-31.)

Every relation of the form $x_0 = \phi(x_1, \ldots, x_n)$, where ϕ is recursive, is arithmetical and we apply complete induction on the rank of ϕ. Let ϕ have rank $s(s > 1)$. ...

$\phi(0, x_2, \ldots, x_n) = \psi(x_2, \ldots, x_n)$
$\phi(k+1, x_2, \ldots, x_n) = \mu[k, \phi(k, x_2, \ldots, x_n), x_2, \ldots, x_n]$
(where ψ, μ have lower rank than s).

... we apply the following procedure: one can express the relation $x_0 = \phi(x_1, \ldots, x_n)$ with the help of the concept "sequence of numbers" (f)[54] in the following manner:

$$x_0 = \phi(x_1, \ldots, x_n) \sim (\exists f)\{f_0 = \psi(x_2, \ldots, x_n) \;\&\; (\forall k)(k < x_1 \rightarrow f_{k+1} = \mu(k, f_k, x_2, \ldots, x_n) \;\&\; x_0 = f_{x_1}\}$$

If $S(y, x_2, \ldots, x_n), T(z, x_1, \ldots, x_{n+1})$ are the arithmetical relations which, according to the inductive hypothesis, are equivalent to $y = \psi(x_2, \ldots, x_n)$, and $z = \mu(x_1, \ldots, x_{n+1})$ respectively, then we have:

$$x_0 = \phi(x_1, \ldots, x_n) \sim (\exists f)\{S(f_0, x_2, \ldots, x_n) \;\&\; (\forall k)[k < x_1 \rightarrow T(f_{k+1}, k, x_2, \ldots, x_n)] \;\&\; x_0 = f_{x_1}\} \quad (17)$$

Now we replace the concept "sequence of numbers" by "pairs of numbers" by correlating with the number pair n, d the sequence of numbers $f^{(n,d)}$ $(f_k^{(n,d)} =$

[54] f denotes here a variable whose domain is the sequence of natural numbers. The $(k+1)$st term of a sequence f is designated f_k (and the first, f_0).

$[n]_{1+(k+1)d}$, where $[n]_p$ denotes the smallest non-negative remainder of n modulo p).

Then:

Lemma 1: If f is an arbitrary sequence of natural numbers and k is an arbitrary natural number, then there exists a pair of natural numbers n, d such that $f^{(n,d)}$ and f coincide in their first k terms.

Proof: Let l be the greatest of the numbers $k, f_0, f_1, \ldots, f_{k-1}$. Determine n so that

$n \equiv f_i \ [mod \ (1 + (i+1)l!)]$ for $i = 0, 1, \ldots, k-1$,

which is possible, since any two of the numbers $1 + (i+1)l!$ $(i = 0, 1, \ldots, k-1)$ are relatively prime. For, a prime dividing two of these numbers must also divide the difference $(i_1 - i_2)l!$ and therefore, since $i_1 - i_2 < l$, must also divide $l!$, which is impossible. The number pair $n, l!$ fulfills our requirement.

Since the relation $x = [n]_p$ is defined by

$x \equiv n \ (mod \ p) \ \& \ x < p$

and is therefore arithmetical, then so also is the relation $P(x_0, x_1, \ldots, x_n)$ defined as follows:

$$P(x_0, x_1, \ldots, x_n) \equiv (\exists n, d)\{S([n]_{d+1}, x_2, \ldots, x_n) \ \& \ (\forall k)[k < x_1 \rightarrow T([n]_{1+d(k+2)}, k, [n]_{1+d(k+1)}, x_2, \ldots, x_n)] \ \& \ x_0 = [n]_{1+d(x_1+1)}\}$$

which, according to (17) and Lemma 1, is equivalent to $x_0 = \phi(x_1, \ldots, x_n)$ (in the sequence f in (17) only its values up to the $(x+1)$th term matter). Thus, Theorem VII(2) is proved.

Note: Gödel's remark that "in the sequence f in (17) only its values up to the $(x+1)$th term matter" is significant. The proof of Lemma 9 depends upon the fact that the equivalence between $f^{(n,d)}$ and f cannot be extended non-terminatingly.

7 Appendix B: Algorithmic verifiability vis à vis algorithmic computability

(Excerpted from [An13].)

Theorem: There are mathematical functions that are algorithmically verifiable but not algorithmically computable.

Proof: (a) Since any real number is mathematically definable as the limit of a Cauchy sequence of rational numbers:

- Let $R(n)$ denote the n^{th} digit in the decimal expression of the real number R in binary notation.

- Then, for any given natural number n, there is an algorithm $AL_{(R,\ n)}$ that can decide the truth/falsity of each proposition in the finite sequence:

  $\{R(1) = 0, R(2) = 0, \ldots, R(n) = 0\}$.

- Hence, for any real number R, the relation $R(x) = 0$ is algorithmically verifiable trivially.

(b) Since it follows from Alan Turing's Halting argument[55] that there are algorithmically uncomputable real numbers:

- Let $[R(n)]$ denote the n^{th} digit in the decimal expression of an algorithmically *uncomputable* real number R in binary notation.

- By (a), the relation $[R(x) = 0]$ is algorithmically verifiable trivially.

- However, by definition there is no algorithm AL_R that can decide the truth/falsity of each proposition in the denumerable sequence:

  $\{[R(1) = 0], [R(2) = 0], \ldots\}$.

- Hence the relation $[R(x) = 0]$ is not algorithmically computable.

(c) We conclude that the relation $[R(x) = 0]$ is algorithmically verifiable but not algorithmically computable. The theorem follows. □

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