Research outline II

Why we shouldn't fault Lucas and Penrose for continuing to believe in the Gödelian argument against computationalism - I

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Most reasoned critiques (e.g., 1996, PSYCHE, 2(23); 2000, Journal of Experimental and Theoretical Artificial Intelligence 12: 307-329; 1993, Behavioral and Brain Sciences, 16, 611-612) of Lucas' and Penrose's well-known Gödelian arguments against computationalism are unassailable. They fail, however, to satisfactorily explain why Lucas and Penrose - reasonable men, both - remain convinced of the essential soundness of their arguments.

One reason could be that Lucas and Penrose have unquestioning faith in, and uncritically follow, standard expositions of classical theory in overlooking what Gödel (1931. On formally undecidable propositions of Principia Mathematica and related systems I. In M. Davis. 1965. The Undecidable. Raven Press. New York. Theorem VI. p24) has implicitly proven; another, that they have similar faith in, and, as uncritically, accept Gödel's (1931, p27) own, informal, interpretation of the implications of this Theorem as definitive.

If so, they should not be taken to task on either count for their faith; it is standard expositions of Gödel's reasoning that are ambiguously silent on both issues.

Specifically, as we show below, the Tarskian satisfaction and Tarskian truth (A. Tarski. 1936. The concept of truth in the languages of the deductive sciences. In Logic, Semantics, Metamathematics, papers from 1923 to 1938. Hackett Publishing Company) of the formulas of an Arithmetic, and the soundness of the Arithmetic itself, can, indeed, be formally defined in an effectively verifiable manner within the Arithmetic.

So, why do such definitions elude serious enquiry by Lucas and Penrose?

One factor could be that Tarski's Theorem (ibid. 1936) appears to implicitly suggest that the intuitive truth, of the formulas of Peano Arithmetic under its standard interpretation, cannot be formalised within the Arithmetic.

Tarski's Theorem: The set of Gödel-numbers of the formulas of any first-order Peano Arithmetic, which are intuitively true in the standard model of the Arithmetic, is not arithmetical.

Lucas and Penrose rely on the implication unquestioningly in their Gödelian arguments.

However, the implication does not withstand scrutiny.

1 Tarski's definitions of satisfiability and truth

To see this, we note the following, standard, definitions of the satisfiability, and truth, of formulas of a formal language, say L, under a well-defined interpretation, say M, due to Tarski (ibid. 1936).

For instance, a formula $[R(x)]$ of L is defined as satisfied under M if, and only if, its corresponding interpretation, say $R(x)$, holds in M for any assignment of a value s that lies within the range of the variable x in M.

Tarski's definitions are mathematically significant only if we assume that, given any s in M, we can decide whether, or not, $R(s)$ holds, or must hold, in M. Where L is PA, the Church-Turing Thesis postulates that such decidability must be algorithmic. Note that, in principle, this can be weakened to effective, instantiational, decidability only - the minimum requirement of Tarski's definitions.

The formula $[(\forall x)R(x)]$ of L is, then, defined as true under the interpretation M if, and only if, $[R(x)]$ is satisfied under M.

Moreover, the formula $[\neg(\forall x)R(x)]$ of L is, further, defined as true under the interpretation M if, and only if, $[(\forall x)R(x)]$ is not true under M.

Clearly, mathematical satisfaction and truth are defined relative only to decidability in an interpretation.

Both, Lucas and Penrose, quite reasonably, therefore, attempt to draw philosophical conclusions from the meta-logical status of the intuitive decidability given in mathematical reasoning to the formulas of Peano Arithmetic under its standard, intuitive, interpretation.

2 Defining formal satisfaction and formal truth verifiably

However, if we take M to be an interpretation of L in L itself, then we have the formalisation of the concepts of verifiable, and unarguable, formal satisfaction, and formal truth, of the formulas $[R(x)]$ and $[(\forall x)R(x)]$ of L, respectively, in L, as:

The formula $[R(x)]$ of L is defined as formally satisfied under L if, and only if $[R(s)]$ is provable in L for any term $[s]$ that can be substituted for the variable $[x]$ in $[R(x)]$.

The formula $[(\forall x)R(x)]$ of L is formally true in L if, and only if, $[R(x)]$ is formally satisfied in L.

3 Defining formal soundness verifiably

If we, further, define formal soundness as the property that the axioms of a theory are satisfied in the theory itself, and that the rules of inference preserve formal truth, then, it follows that the theorems of any formally sound theory are formally true in the theory.

It is straightforward to verify that first-order Peano Arithmetic is, indeed, formally sound.

4 Gödelian propositions

Now, even if the formula $[(\forall x)R(x)]$ is not provable in L, it would be formally true in L if, and only if, the formula $[R(s)]$ were provable in L for any well-defined term $[s]$ of L that could be substituted for $[x]$ in $[R(x)]$.

The existence of such a, Gödelian, proposition is, precisely, what Gödel (1931, p24) proves in his Theorem VI for a consistent Peano Arithmetic. He constructs a formula, $[(\forall x)R(x)]$, of PA that is, itself, unprovable in PA, even though, for any given numeral $[n]$, $[R(n)]$ is provable in PA.

So, Gödel has constructed a formally unprovable Arithmetical formula that is not only intuitively true in the standard, intuitive, interpretation of the Arithmetic, but which is also formally true in the Arithmetic in a verifiable, and intuitionistically unobjectionable, manner that leaves no room for dispute as to its 'truth' status vis--vis the PA axioms!

Moreover, since the Arithmetic can be shown to be formally sound - again in a verifiable, and intuitionistically unobjectionable, manner - we no longer need appeal to the arguable assumption that the Arithmetic is intuitively sound under the standard interpretation. PA is said to be sound for a class S of sentences if, whenever PA proves $[\phi]$ with $[\phi]$ in S, then ϕ is true in the structure N of natural numbers.

Replacing the, philosophically debatable, concepts of intuitive decidability with, constructively verifiable, definitions of formal decidability should, thus, place Lucas' and Penrose's Gödelian arguments in better perspective.

5 Defining arithmetic truth verifiably

The significance of the above remarks is highlighted by the following.

Let $[F(x_1, \ldots, x_n)]$ be a PA-formula whose Gödel-number in Gödel's notation[1] is f, and $F(x_1, \ldots, x_n)$ its standard interpretation under which[2]:

[1] [Go31], p13.

[2] Excerpted from [Me64], p.107. However, curiously, standard texts do not recognise that—as I show in [An08f]—the interpretation of formal existential quantification such as '$[(\exists x)R(x)]$'—which is merely an abbreviation for the formula '$[\neg(\forall x)\neg R(x)]$'—also needs to be specifically defined under the 'standard' interpretation of PA in terms of Hilbert's ϵ-function.

Thus, although Wang ([Wa63], p.314-315) highlights the significance of Hilbert's ϵ-function, he implicitly implies that any interpretation of the quantifiers of a formal language are to be determined by Hilbert's definitions of quantification—by means of the ϵ-function—in the underlying predicate calculus that is to be applied in common to all the mathematical languages under consideration:

"...it is customary to take for granted a basic logic of inference (the theory of quantification or the predicate calculus) which deals with the logical constants "if-then", "not", "all", "some", "or", "and", "if and only if". These are, as we know, standard axiom systems for quantification theory. If we adjoin one such system to an axiom system for geometry, we get a more thoroughly formalized system."

In other words, formal quantification cannot be qualified further under an interpretation of a language once the underlying predicate calculus has been prescribed.

This is a crucial assumption that is echoed in current literature, which seem to treat the issue

(a) the set of non-negative integers is the domain,

(b) the integer 0 is the interpretation of the symbol 0,

(c) the successor operation (addition of 1) is the interpretation of the $'$ function (i.e., of f_1^1,

(d) ordinary addition and multiplication are the interpretations of $+$ and $.$,

(e) the interpretation of the predicate letter $=$ is the identity relation.

Then, following Gödel[3], we can define the concept 'f is a PROVABLE FORMULA of PA' as:

$\text{Bew}_{PA}(f) \equiv (\exists y) y\, B_{PA}\, f$

Now, we can also define the concept 'f is a TRUE FORMULA under a sound interpretation $\mathcal{I}$ of PA' in the above notation as (where p_n denotes the n^{th} prime number):

$$\text{Tr}_{PA}(f) \equiv (\forall u_1) \ldots (\forall u_n)(\exists y) y\, B\, \{Sb_{PA}(f \begin{smallmatrix} 17 \\ Z(u_1) \end{smallmatrix} \ldots \begin{smallmatrix} p_{n+7} \\ Z(u_n) \end{smallmatrix})\}$$

In other words, $\text{Tr}_{PA}(f)$ holds if, and only if, for any given natural number sequence $a_1, \ldots, a_n$, the PA-formula $[F(a_1, \ldots, a_n)]$ is PA-provable.

Hence, if $\text{Tr}_{PA}(f)$ holds then $[F(x_1, \ldots, x_n)]$ is a true formula under any sound interpretation of PA.

Now Gödel has also shown that there are arithmetical relations $R(y)$ and $S(u_1, \ldots, u_n)$ such that:

(i) for any natural number a:

$a\, B_{PA}\, f \equiv R(a)$

(ii) for any natural number sequence $a_1, \ldots, a_n$:

$$y\, B\, \{Sb_{PA}(f \begin{smallmatrix} 17 \\ Z(a_1) \end{smallmatrix} \ldots \begin{smallmatrix} p_{n+7} \\ Z(a_n) \end{smallmatrix})\} \equiv S(a_1, \ldots, a_n)$$

Hence, contrary to the current interpretations of Tarski's and Gödel's formal reasoning, both PA-provability and PA-Truth (under a sound interpretation) are 'arithmetical'.

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at best implicitly—and otherwise vaguely—as an intuitionistically unobjectionable feature of Tarski's standard semantic definitions [Ta33] of the satisfaction and truth of the formulas of a formal language under an interpretation $\mathcal{I}$ (see [Me64], p.52, V(ii); [Sc67], p.13 & p.23, Ex(1); [BBJ03], pp.101-102 & p.104(9.2); [EC89], pp.174 & 176).

However, I show in [An08f] why the interpretation of formal existential quantification in a particular case need not follow Tarski's standard (but patently non-constructive since they are essentially unverifiable objectively) semantic definitions of truth; why it needs to be made explicit for each interpretation of a formal language; and why, where a sound interpretation of PA is concerned, if the PA-formula $[R(x)]$ interprets as the arithmetic relation denoted by $R'(x)$, the PA-formula $[(\exists\ x)R(x)]$ cannot be assumed to interpret as the arithmetical proposition '$R'(x)$ holds for some natural number x'—which is symbolically denoted by its abbreviation '$(\exists x)R'(x)$'.

[3] [Go31], p.22(Dfn.46).

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