

The significance of Aristotle's particularisation in the foundations of mathematics, logic and computability I

Cohen and the Axiom of Choice

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Abstract

The logic underlying our current interpretations of all first-order formal languages—which provide the formal foundations for all computing languages—is Aristotle's logic of predicates. I show, first, that a fundamental tenet of this logic, namely Aristotelean particularisation, is a subjective, and objectively unverifiable, postulation that is 'stronger' than the Axiom of Choice; and that, second, any putative model of ZF which appeals to Aristotelean particularisation—under Tarski's definitions of what constitute's an interpretation of the language—is not sound.¹

1 Introduction

Classical theory—following the 2000-year old philosophical perspective of Greek philosophers such as Aristotle—asks only that the 'core' axioms of a formal language, and its rules of inference, be interpretable—under Tarski's definitions of the satisfiability, and truth, of the formulas of a formal language under an interpretation—as self-evidently sound² propositions.

The subjective—and essentially irresolvable—element in the 'soundness' of such a viewpoint is self-evident.

In these investigations, I highlight the limitations of such subjectivity, and, in the case of the Peano Arithmetic, PA, show how to avoid them by requiring that the 'core' axioms of PA, and its rules of inference, be interpretable as algorithmically (and, ipso facto, objectively) verifiable 'sound' propositions, and consider some consequences.

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²The word 'sound' is occasionally used informally as a synonym for 'reliable', but mostly as the technical term of mathematical logic defined precisely in the next section. The distinction is assumed to be obvious from the context in which it is used.

1.1 Preliminary Definitions

Aristotelean particularisation: This holds that an assertion such as, ‘There exists an unspecified x such that $F(x)$ holds’—usually denoted symbolically by ‘ $(\exists x)F(x)$ ’—can always be validly inferred in the classical, Aristotelean, logic of predicates³ from the assertion, ‘It is not the case that, for any given x , $F(x)$ does not hold’—usually denoted symbolically by ‘ $\neg(\forall x)\neg F(x)$ ’.

Simple consistency: A formal system S is simply consistent if, and only if, there is no S -formula $[F(x)]$ for which both $[(\forall x)F(x)]$ and $[\neg(\forall x)F(x)]$ are S -provable.

ω -consistency: A formal system S is ω -consistent if, and only if, there is no S -formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S -provable and, second, $[F(a)]$ is S -provable for any given S -term $[a]$.

Soundness (formal system): A formal system S is sound under an interpretation \mathcal{I}_S if, and only if, every theorem $[T]$ of S translates as ‘ $[T]$ is true under \mathcal{I}_S ’.

Soundness (interpretation): An interpretation \mathcal{I}_S of a formal system S is sound if, and only if, S is sound under the interpretation \mathcal{I}_S .

Soundness in classical logic. In classical logic, a formal system S is defined as ‘sound’ if, and only if, it has an interpretation; and an interpretation is defined as the assignment of meanings to the symbols, and truth-values to the sentences, of the formal system. Moreover, any such interpretation is a model of the formal system. This definition suffers, however, from an implicit circularity: the formal logic L underlying any interpretation of S is implicitly assumed to be ‘sound’. The above definitions seek to avoid this implicit circularity by delinking the defined ‘soundness’ of a formal system under an interpretation from the implicit ‘soundness’ of the formal logic underlying the interpretation. This admits the case where, even if L_1 and L_2 are implicitly assumed to be sound, $S + L_1$ is sound, but $S + L_2$ is not. Moreover, an interpretation of S is now a model for S if, and only if, it is sound.⁴

Axiom of Choice (a standard interpretation): Given any set S of mutually disjoint non-empty sets, there is a set C containing a single member from each element of S .

2 Hilbert’s formalisation of Aristotle’s logic of predicates

Now, a fundamental tenet of classical logic—unrestrictedly adopted by formal first-order predicate calculus as axiomatic⁵—is Aristotelean particularisation.

In a 1927 address, Hilbert reviewed, as part of his ‘proof theory’, his axiomatisation \mathcal{L}_ε of classical Aristotle’s logic of predicates as a formal first-order ε -predicate calculus⁶.

A specific aim of the axiomatisation appears to have been the introduction of a primitive choice-function symbol, ‘ ε ’, for formalising the existence of the unspecified object in Aristotle’s particularisation⁷:

³[HA28], pp.58-59.

⁴My thanks to Professor Rohit Parikh for highlighting the need for making such a distinction explicit.

⁵See [Hi25], p.382; [HA28], p.48; [Sk28], p.515; [Be59], pp.178 & 218; [Co66], p.4.

⁶[Hi27], pp.465-466.

⁷[Hi25], p.382; [Ca62], p.156; see also Appendix A.

“... $\varepsilon(A)$ stands for an object of which the proposition $A(a)$ certainly holds if it holds of any object at all ...”⁸

Hilbert showed, moreover, how the quantifiers ‘ \forall ’ and ‘ \exists ’ are definable using the choice-function ‘ ε ’ (see Appendix A)—and noted that⁹:

“... The fundamental idea of my proof theory is none other than than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds.”

More precisely¹⁰:

Lemma 1 *\mathcal{L}_ε adequately expresses—and yields, under a suitable interpretation—Aristotle’s logic of predicates if the ε -function is interpreted so as to yield the unspecified object in Aristotelean particularisation.*

What came to be known later as Hilbert’s Program¹¹—which was built upon Hilbert’s ‘proof theory’—can be viewed as, essentially, the subsequent attempt to show that the formalisation was also necessary for communicating Aristotle’s logic of predicates effectively and unambiguously under any interpretation of the formalisation. This goal is implicit in Hilbert’s remarks¹²:

“Mathematics in a certain sense develops into a tribunal of arbitration, a supreme court that will decide questions of principle—and on such a concrete basis that universal agreement must be attainable and all assertions can be verified.”

“... a theory by its very nature is such that we do not need to fall back upon intuition or meaning in the midst of some argument.”

2.1 The postulation of an ‘unspecified’ object in Aristotle’s particularisation is ‘stronger’ than the Axiom of Choice

The difficulty in attaining this goal constructively along the lines desired by Hilbert—in the sense of the above quotes—becomes evident from Rudolf Carnap’s analysis in a 1962 paper, “On the use of Hilbert’s ε -operator in scientific theories”¹³. From this it follows that, if we define a formal language ZF_ε by replacing:

$$\begin{aligned} [(\forall x)F(x)] & \text{ with } [F(\varepsilon_x(\neg F(x)))] \\ [(\exists x)F(x)] & \text{ with } [F(\varepsilon_x(F(x)))] \end{aligned}$$

in the Zermelo-Fraenkel set theory ZF, then:

Lemma 2 *The Axiom of Choice is true in any sound interpretation of the Zermelo-Fraenkel set theory ZF_ε that admits Aristotle’s logic of predicates.*

Thus, the postulation of an ‘unspecified’ object in Aristotelean particularisation is a stronger postulation than the Axiom of Choice!

⁸Note that $\varepsilon(A)$ need not be a ‘term’ of \mathcal{L}_ε , since it is a term if, and only if, $A(a)$ ‘holds’ for some term a .

⁹[Hi27], p.475.

¹⁰cf. [Hi25], pp.382-383; [Hi27], p.466(1).

¹¹See, for instance, the Stanford Encyclopedia of Philosophy: Hilbert’s Program.

¹²[Hi25], p.384; [Hi27], p.475.

¹³[Ca62], pp.157-158; see also Wang’s remarks [Wa63], pp.320-321.

2.1.1 Cohen and The Axiom of Choice

The significance of this is seen in the accepted interpretation of Cohen’s argument in his 1963-64 papers¹⁴; the argument is accepted as definitively establishing that the Axiom of Choice is essentially independent of a set theory such as ZF.

Now, Cohen’s argument—in common with the arguments of many important theorems in standard texts on the foundations of mathematics and logic—appeals to the unspecified object in Aristotle’s particularisation when interpreting the existential axioms of ZF (or statements about ZF ordinals).

This is seen in his proof¹⁵ and application of the—seemingly paradoxical¹⁶—(downwards) Löwenheim-Skolem Theorem.

(Downwards) Löwenheim-Skolem Theorem¹⁷: If a first-order proposition is satisfied in any domain at all, then it is already satisfied in a denumerably infinite domain.

Cohen appeals to this theorem for legitimising putative models of a language—such as the standard model ‘M’ of ZF¹⁸, and its forced derivative ‘N’¹⁹—in his argument²⁰.

Now, the significance of Hilbert’s formalisation of Aristotle’s particularisation by means of the ε -function is seen in Cohen’s following remarks, where he explicitly appeals in the above argument to a semantic—rather than formal—definition of the unspecified object in Aristotle’s particularisation²¹:

“When we try to construct a model for a collection of sentences, each time we encounter a statement of the form $(\exists x)B(x)$ we must invent a symbol \bar{x} and adjoin the statement $B(\bar{x})$ when faced with $(\exists x)B(x)$, we should choose to have it false, unless we have already invented a symbol \bar{x} for which we have strong reason to insist that $B(\bar{x})$ be true.”

Cohen, then, shows that:

Lemma 3 *The Axiom of Choice is false in N.*

2.1.2 Any interpretation of ZF which appeals to Aristotle’s particularisation is not sound

Since Hilbert’s ε -function formalises precisely Cohen’s concept of ‘ \bar{x} ’—more properly, ‘ \bar{x}_B ’—as $[\varepsilon_x B(x)]$, it follows that:

Theorem 1 *Any model of ZF, in which the quantifiers are interpreted according to Aristotle’s logic of predicates, is a model of ZF $_\varepsilon$ if the expression $[\varepsilon_x B(x)]$ is interpreted to yield Cohen’s symbol ‘ \bar{x}_B ’ whenever $[B(\varepsilon_x(B(x)))]$ interprets as true.*

¹⁴[Co63] & [Co64].

¹⁵[Co66], p.19.

¹⁶See Skolem’s remarks [Sk22], p.295; also [Co66], p.19.

¹⁷[Lo15], p.245, Theorem 6; [Sk22], p.293.

¹⁸[Co66], p.19 & p.82.

¹⁹[Co66], p.121.

²⁰[Co66], p.83 & p.112-118.

²¹[Co66], p.112; see also p.4.

Hence Cohen’s argument is also applicable to ZF_ε . However, since the Axiom of Choice is true in any sound interpretation of ZF_ε which appeals to Aristotle’s logic of predicates, Cohen’s argument²²—when applied to ZF_ε —actually shows that:

Corollary 1 *ZF_ε has no model that appeals to Aristotle’s logic of predicates.*

Corollary 2 *ZF has no model that appeals to Aristotle’s particularisation.*

We cannot, therefore, conclude that the Axiom of Choice is essentially independent of the axioms of ZF , since none of the putative models ‘forced’ by Cohen (in his argument for such independence) are defined by a sound interpretation of ZF .

3 Cohen and the Gödelian Argument

At the conclusion of his lectures on “Set Theory and the Continuum Hypothesis”, delivered at Harvard University in the spring term of 1965, Cohen remarked²³:

“We close with the observation that the problem of CH is not one which can be avoided by not going up in type to sets of real numbers. A similar undecidable problem can be stated using only the real numbers. Namely, consider the statement that every real number is constructible by a countable ordinal. Instead of speaking of countable ordinals we can speak of suitable subsets of ω . The construction $\alpha \rightarrow F_\alpha$ for $\alpha \leq \alpha_0$, where α_0 is countable, can be completely described if one merely gives all pairs (α, β) such that $F_\alpha \in F_\beta$. This in turn can be coded as a real number if one enumerates the ordinals. In this way one only speaks about real numbers and yet has an undecidable statement in ZF . One cannot push this farther and express any of the set-theoretic questions that we have treated as statements about integers alone. Indeed one can postulate as a rather vague article of faith that any statement in arithmetic is decidable in “normal” set theory, i.e., by some recognizable axiom of infinity. This is of course the case with the undecidable statements of Gödel’s theorem which are immediately decidable in higher systems.”

Cohen appears to assert here that if ZF is consistent, then we can ‘see’ that the Continuum Hypothesis is subjectively true for the integers under some model of ZF , but—along with the Generalised Continuum Hypothesis—we cannot objectively ‘assert’ it to be true for the integers since it is not provable in ZF , and hence not true in all models of ZF .

However, by this argument, Gödel’s undecidable arithmetical propositions, too, can be ‘seen’ to be subjectively true for the integers in the standard model of PA , but cannot be ‘asserted’ to be true for the integers since the statements are not provable in an ω -consistent PA , and hence they are not true in all models of an ω -consistent PA !

²²[Co63] & [Co64]; [Co66].

²³[Co66], p.151.

The latter is, essentially, John Lucas' well-known Gödelian argument²⁴, forcefully argued by Roger Penrose in his popular expositions, 'Shadows of the Mind'²⁵ and 'The Emperor's New Mind'²⁶.

As I have argued in *The Reasoner*²⁷, the argument is plausible, but unsound. It is based on a misinterpretation—of what Gödel actually proved formally in his 1931 paper—for which, moreover, neither Lucas nor Penrose ought to be taken to account²⁸.

The distinction sought to be drawn by Cohen is curious, since we have shown that his argument—which assumes that sound interpretations of ZF can appeal to Aristotle's particularisation—actually establishes that sound interpretations of ZF cannot appeal to Aristotle's particularisation; just as I show in a companion paper²⁹ that Gödel's argument³⁰ actually establishes that any sound interpretation of PA, too, cannot appeal to Aristotle's particularisation.

Loosely speaking, the cause of the undecidability of the Continuum Hypothesis—and of the Axiom of Choice—in ZF as shown by Cohen, and that of Gödel's³¹ undecidable proposition in Peano Arithmetic, is common; it is interpretation of the existential quantifier under an interpretation as Aristotelean particularisation.

In Cohen's case, such interpretation is made explicitly and unrestrictedly in the underlying predicate logic³² of ZF, and in its interpretation in Aristotle's logic of predicates³³.

In Gödel's case it is made explicitly—but formally to avoid attracting intuitionistic objections—through his specification of what he believed to be a 'much weaker assumption'³⁴ of ω -consistency for his formal system P of Peano Arithmetic³⁵.

4 Appendix A: Hilbert's interpretation of quantification

Hilbert interpreted quantification in terms of his ε -function as follows³⁶:

IV. The logical ε -axiom

13. $A(a) \rightarrow A(\varepsilon(A))$

²⁴[Lu61].

²⁵[Pe94].

²⁶[Pe90].

²⁷[An07a]; [An07b]; [An07c].

²⁸[An07b]; [An07c].

²⁹[An09a].

³⁰In [Go31], p.24, Theorem VI.

³¹Also Rosser's; see [An09b].

³²[Co66], p.4.

³³[Co66] p.112.

³⁴I highlight the significance of Gödel's 'much weaker assumption' in [An09a], where I note that, if a Peano Arithmetic has an interpretation that appeals to Aristotle's particularisation—such as the interpretation currently defined as the 'standard' interpretation of any Peano Arithmetic—and if the interpretation is sound, then the Arithmetic is ω -consistent.

³⁵[Go31], p.9 & pp.23-24.

³⁶[Hi27], p.466.

Here $\varepsilon(A)$ stands for an object of which the proposition $A(a)$ certainly holds if it holds of any object at all; let us call ε the logical ε -function. ...

1. By means of ε , “all” and “there exists” can be defined, namely, as follows:

$$(i) (\forall a)A(a) \leftrightarrow A(\varepsilon(\neg A))$$

$$(ii) (\exists a)A(a) \leftrightarrow A(\varepsilon(A)) \dots$$

On the basis of this definition the ε -axiom IV(13) yields the logical relations that hold for the universal and the existential quantifier, such as:

$$(\forall a)A(a) \rightarrow A(b) \dots (\text{Aristotle's dictum}),$$

and:

$$\neg((\forall a)A(a)) \rightarrow (\exists a)(\neg A(a)) \dots (\text{principle of excluded middle}).$$

Thus, Hilbert's interpretation of universal quantification — defined in (i) — is that the sentence $(\forall x)F(x)$ holds (*under a consistent interpretation \mathcal{I}*) if, and only if, $F(a)$ holds whenever $\neg F(a)$ holds for any given a (*in \mathcal{I}*); hence $\neg F(a)$ does not hold for any a (*since \mathcal{I} is consistent*), and so $F(a)$ holds for any given a (*in \mathcal{I}*).

Further, Hilbert's interpretation of existential quantification — defined in (ii) — is that $(\exists x)F(x)$ holds (*in \mathcal{I}*) if, and only if, $F(a)$ holds for some a (*in \mathcal{I}*).

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