The significance of Aristotle’s particularisation in the foundations of mathematics, logic and computability II

Gödel and formally undecidable arithmetical propositions

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Abstract

The logic underlying our current interpretations of all first-order formal languages—which provide the formal foundations for all computing languages—is Aristotle’s logic of predicates. I show, first, that a fundamental tenet of this logic, namely Aristotlean particularisation, is a subjective, and objectively unverifiable, postulation that is ‘stronger’ than the Axiom of Choice; and that, second, ‘standard’ interpretations of languages such as ZF or PA—which appeal to Aristotlean particularisation under Tarski’s definitions of what constitute’s an interpretation of the language—are not ‘sound’.

1 Introduction

Classical theory—following the 2000-year old philosophical perspective of Greek philosophers such as Aristotle—asks only that the ‘core’ axioms of a formal language, and its rules of inference, be interpretable—under Tarski’s definitions of the satisfiability, and truth, of the formulas of a formal language under an interpretation—as self-evidently sound propositions.

The subjective—and essentially irresolvable—element in the ‘soundness’ of such a viewpoint is self-evident.

In these investigations, I, first, highlight the limitations of such subjectivity, and, in the case of the Peano Arithmetic, PA, show how to avoid them by requiring that the ‘core’ axioms of PA, and its rules of inference, be interpretable as algorithmically (and, ipso facto, objectively) verifiable true propositions, and consider some consequences.

Keywords: Aristotle, Gödel, ω-consistent, particularisation, Tarski.

1 The word ‘sound’ is occasionally used informally as a synonym for ‘reliable’, but mostly as the technical term of mathematical logic defined precisely in the next section. The distinction is assumed to be obvious from the context in which it is used.
1.1 Preliminary Definitions and Comments

Aristotelian particularisation: This holds that an assertion such as, ‘There exists an unspecified $x$ such that $F(x)$ holds’—usually denoted symbolically by ‘$(\exists x)F(x)$’—can always be validly inferred in the classical, Aristotelian, logic of predicates from the assertion, ‘It is not the case that, for any given $x$, $F(x)$ does not hold’—usually denoted symbolically by ‘$\sim(\forall x)\sim F(x)$’.

Notation: I shall henceforth use square brackets to indicate that the contents represent a symbol or a formula of a formal theory, generally assumed to be well-formed unless otherwise indicated by the context.

In other words, expressions inside the square brackets are to be only viewed syntactically as juxtaposition of symbols that are to be formed and manipulated upon strictly in accordance with specific rules for such formation and manipulation—in the manner of a mechanical or electronic device—without any regards to what the symbolism might represent semantically under an interpretation that gives them meaning.

Moreover, even though the formula ‘$[R(x)]$’ of a formal Arithmetic may interpret as the arithmetical relation expressed by ‘$R^*(x)$’, the formula ‘$(\exists x)R(x)$’ need not interpret as the arithmetical proposition denoted by the abbreviation ‘$(\exists x)R^*(x)$’. The latter denotes the phrase ‘There is some $x$ such that $R^*(x)$’. As Brouwer had noted[[2]], this concept is not always capable of an unambiguous meaning that can be represented in a formal language by the formula ‘$(\exists x)[R(x)]$’.

By ‘expressed’ I mean here that the symbolism is simply a short-hand abbreviation for referring to abstract concepts that may, or may not, be capable of a precise ‘meaning’. Amongst these are symbolic abbreviations which are intended to express the abstract concepts—particularly those of ‘existence’—involved in propositions that refer to non-terminating processes and infinite aggregates.

Provability: A formula $[F]$ of a formal system $S$ is provable in $S$ ($S$-provable) if, and only if, there is a finite sequence of $S$-formulas $[F_1],[F_2],...,[F_n]$ such that $[F_n]$ is $[F]$ and, for all $1 \leq i \leq n$, $[F_i]$ is either an axiom of $S$ or a consequence of the axioms of $S$, and the formulas preceding it in the sequence, by means of the rules of deduction of $S$.

The structure $\mathcal{N}$: The structure of the natural numbers—namely, $\mathcal{N}$ (the set of natural numbers); $=$ (equality); ‘$\prime$’ (the successor function); $+$ (the addition function); $\ast$ (the product function); $0$ (the null element).

The axioms of first-order Peano Arithmetic (PA)

**PA$_1$:** $[(x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))]$;

**PA$_2$:** $[(x_1 = x_2) \rightarrow (x_1' = x_2')]$;

**PA$_3$:** $[0 \neq x_1']$;

**PA$_4$:** $[(x_1' = x_2') \rightarrow (x_1 = x_2)]$;

**PA$_5$:** $[(x_1 + 0) = x_1]$;

**PA$_6$:** $[(x_1 + x_2') = (x_1 + x_2)']$;

**PA$_7$:** $[(x_1 \ast 0) = 0]$;

**PA$_8$:** $[(x_1 \ast x_2') = ((x_1 \ast x_2) + x_1)]$;

**PA$_9$:** For any well-formed formula $[F(x)]$ of PA:

$$[F(0) \rightarrow (\forall x)(F(x) \rightarrow F(x'))] \rightarrow (\forall x)F(x)].$$
Generalisation in PA: If \([A]\) is PA-provable, then so is \((\forall x)A\).

Modus Ponens in PA: If \([A]\) and \([A \to B]\) are PA-provable, then so is \([B]\).

Standard interpretation of PA: The standard interpretation \(I_{PA(Standard/Tarski)}\) of PA over the structure \(N\) is the one in which the logical constants have their ‘usual’ interpretations in Aristotle’s logic of predicates, and

- (a) the set of non-negative integers is the domain;
- (b) the integer 0 is the interpretation of the symbol \([0]\);
- (c) the successor operation (addition of 1) is the interpretation of the \([\ast]\)
  function;
- (d) ordinary addition and multiplication are the interpretations of \([+]\) and \([\ast]\);
- (e) the interpretation of the predicate letter \([=]\) is the identity relation.

Simple consistency: A formal system \(S\) is simply consistent if, and only if, there is no \(S\)-formula \([F(x)]\) for which both \([(\forall x)F(x)]\) and \([\neg(\forall x)F(x)]\) are \(S\)-provable.

\(\omega\)-consistency: A formal system \(S\) is \(\omega\)-consistent if, and only if, there is no \(S\)-formula \([F(x)]\) for which, first, \([\neg(\forall x)F(x)]\) is \(S\)-provable and, second, \([F(a)]\) is \(S\)-provable for any given \(S\)-term \([a]\).

Soundness (formal system): A formal system \(S\) is sound under an interpretation \(I_S\) if, and only if, every theorem \([T]\) of \(S\) translates as ‘\([T]\) is true under \(I_S\)’.

Soundness (interpretation): An interpretation \(I_S\) of a formal system \(S\) is sound if, and only if, \(S\) is sound under the interpretation \(I_S\).

Soundness in classical logic. In classical logic, a formal system \(S\) is defined as ‘sound’ if, and only if, it has an interpretation; and an interpretation is defined as the assignment of meanings to the symbols, and truth-values to the sentences, of the formal system. Moreover, any such interpretation is a model of the formal system. This definition suffers, however, from an implicit circularity: the formal logic \(L\) underlying any interpretation of \(S\) is implicitly assumed to be ‘sound’. The above definitions seek to avoid this implicit circularity by delinking the defined ‘soundness’ of a formal system under an interpretation from the implicit ‘soundness’ of the formal logic underlying the interpretation. This admits the case where, even if \(L_1\) and \(L_2\) are implicitly assumed to be sound, \(S + L_1\) is sound, but \(S + L_2\) is not. Moreover, an interpretation of \(S\) is now a model for \(S\) if, and only if, it is sound.

Categoricity: A formal system \(S\) is categorical if, and only if, it has a sound interpretation and any two sound interpretations of \(S\) are isomorphic.

2 Hilbert’s epsilon-predicate calculus formalises Aristotle’s logic of predicates

Now, in a 1927 address, David Hilbert reviewed in detail his axiomatisation \(L_\varepsilon\) of classical Aristotlean predicate logic as a formal first-order \(\varepsilon\)-predicate calculus—\(\varepsilon\), in which he used a primitive choice-function symbol, \([\varepsilon]\), for defining

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6 See [Me64], p.49.
7 See [Me64], p.107.
8 My thanks to Professor Rohit Parikh for highlighting the need for making such a distinction explicit.
9 Compare [Me64], p.91.
10 [Hi27], pp.465-466.
11 See [Hi25], p.382.
the quantifiers $\forall$ and $\exists$—and, essentially, noted that:

**Lemma 1** $\mathcal{L}_\varepsilon$ adequately expresses—and yields, under a suitable interpretation—Aristotle’s logic of predicates if the $\varepsilon$-function is interpreted to yield Aristotelian particularisation.

### 2.1 The postulation of an ‘unspecified’ object in Aristotle’s particularisation is ‘stronger’ than the Axiom of Choice

Further, it follows from Rudolf Carnap’s analysis in a 1962 paper, “On the use of Hilbert’s $\varepsilon$-operator in scientific theories” that, if we define a formal language ZF by replacing:

- $[(\forall x)F(x)]$ with $[F(\varepsilon x(\neg F(x)))]$
- $[(\exists x)F(x)]$ with $[F(\varepsilon x(F(x)))]$

in the Zermelo-Fraenkel set theory ZF, then:

**Lemma 2** The Axiom of Choice is true in any sound interpretation of the Zermelo-Fraenkel set theory ZF, that admits Aristotle’s logic of predicates.

### 2.1.1 Cohen and The Axiom of Choice

Now, in his 1963-64 papers—establishing that the Axiom of Choice is essentially independent of a set theory such as ZF—Paul J. Cohen appeals to Aristotle’s particularisation when interpreting the existential axioms of ZF (or statements about ZF ordinals) for legitimising putative models of a language.

Cohen’s assumption of Aristotle’s particularisation: “When we try to construct a model for a collection of sentences, each time we encounter a statement of the form $(\exists x)B(x)$ we must invent a symbol $\varepsilon$ and adjoin the statement $B(\varepsilon)$. …when faced with $(\exists x)B(x)$, we should choose to have it false, unless we have already invented a symbol $\varepsilon$ for which we have strong reason to insist that $B(\varepsilon)$ be true.”

Cohen, then, shows that:

**Lemma 3** The Axiom of Choice is false in $N$.

### 2.1.2 Any interpretation of ZF which appeals to Aristotle’s particularisation is not sound

Since Hilbert’s $\varepsilon$-function formalises precisely Cohen’s concept of ‘$\varepsilon$’—more properly, ‘$\varepsilon_B$’—as $[\varepsilon_B B(x)]$, it follows that:

**Theorem 1** Any model of ZF, in which the quantifiers are interpreted according to Aristotle’s logic of predicates, is a model of ZF if the expression $[\varepsilon_B B(x)]$ is interpreted to yield Cohen’s symbol ‘$\varepsilon_B$’ whenever $[B(\varepsilon_B(B(x)))]$ interprets as true.
Hence Cohen’s argument is also applicable to ZF\(\varepsilon\). However, since the Axiom of Choice is true in any sound interpretation of ZF\(\varepsilon\) which appeals to Aristotle’s logic of predicates, Cohen’s argument\(^{17}\)—when applied to ZF\(\varepsilon\)—actually shows that:

**Theorem 2** ZF\(\varepsilon\) has no sound interpretation that appeals to Aristotle’s particularisation.\(\Box\)

**Corollary 1** ZF has no sound interpretation that appeals to Aristotle’s particularisation.\(\Box\)

3 Gödel and formally undecidable arithmetical propositions

I now show that Kurt Gödel’s formal reasoning in his 1931 paper\(^{18}\) actually implies that PA is \(\omega\)-inconsistent; and that, as a consequence, the standard interpretation \(\mathcal{I}_{PA}\)(Standard/Tarski) of PA—which, too, appeals to Aristotle’s particularisation—is also not sound.

### 3.1 Consequences of favouring Tarski’s interpretation

Under Tarski’s definitions of the satisfaction, and truth, of the propositions of a language under an interpretation\(^{19}\) the formally defined logical constant \(['∃]'\) in an occurrence such as \(['(∃x)\ldots]'\)—which is formally defined in terms of the primitive (undefined) logical constant \(['∀]' as \(['\neg(∀x)\neg\ldots]'\)—sometimes appeals to an interpretation such as ‘There is some unspecified \(x\) such that . . . ’ in any sound interpretation of any formal first-order mathematical language\(^{20}\).

In other words:

**Lemma 4** If the first-order predicate calculus of a first-order mathematical language admits quantification, then any putative model of the language must interpret existential quantification as Aristotle’s particularisation under Tarski’s definitions of the satisfaction, and truth, of the formulas of a first-order language under an interpretation.\(\Box\)

We thus have:

**Lemma 5** If Aristotle’s particularisation is logically valid, then the standard interpretation \(\mathcal{I}_{PA}\)(Standard/Tarski) of PA is sound.\(\Box\)

**Lemma 6** If \(\mathcal{I}_{PA}\)(Standard/Tarski) is sound, then PA is \(\omega\)-consistent.\(\Box\)

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\(^{17}\) [Co63] \& [Co64]: [Co66].

\(^{18}\) [Go31].

\(^{19}\) [Ta33].

\(^{20}\) See, for instance, [Me64], p.52(ii).
3.2 The significance of omega-consistency: Hilbert’s program

As part of his program for giving mathematical reasoning a finitary foundation, Hilbert proposed an $\omega$-rule as a finitary means of extending a Peano Arithmetic—such as Gödel’s formal Peano Arithmetic $P^{22}$—to a possible completion (i.e. to logically showing that, given any arithmetical proposition, either the proposition, or its negation, is formally provable from the axioms and rules of inference of the extended Arithmetic).

**Hilbert’s $\omega$-Rule:** If it is proved that the $P$-formula $[F(x)]$ interprets as a true numerical formula for each given $P$-numeral $[x]$, then the $P$-formula $[(\forall x)F(x)]$ may be admitted as an initial formula (axiom) in $P$.

We thus have:

**Lemma 7** If we meta-assume Hilbert’s $\omega$-rule for $P$, then a consistent $P$ is necessarily $\omega$-consistent.□

3.3 Aristotle’s particularisation implies PA has non-standard models

Now, in his seminal 1931 paper Gödel showed that:

**Lemma 8** If a Peano Arithmetic $P$ is $\omega$-consistent, then there is a constructively definable $P$-formula $[R(x)]$ such that neither $[(\forall x)R(x)]$ nor $[\neg(\forall x)R(x)]$ are $P$-provable.

Gödel concluded that:

**Lemma 9** Any $\omega$-consistent Peano Arithmetic $P$ has a consistent, but $\omega$-inconsistent, extension $P'$, obtained by adding the formula $[\neg(\forall x)R(x)]$ as an axiom to $P$.□

3.4 A consistent PA is not omega-consistent

However, Gödel also showed that:

**Lemma 10** If $P$ is consistent and $[(\forall x)R(x)]$ is assumed $P$-provable, then $[\neg(\forall x)R(x)]$ is $P$-provable.

By Gödel’s definition of $P$-provability, it follows that:

**Lemma 11** There is a finite sequence $[F_1], \ldots, [F_n]$ of $P$-formulas such that $[F_1]$ is $[(\forall x)R(x)]$, $[F_n]$ is $[\neg(\forall x)R(x)]$, and, for $2 \leq i \leq n$, $[F_i]$ is either a $P$-axiom or a logical consequence of the preceding formulas in the sequence by the rules of inference of $P$.□

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21cf. [Hi30], pp.485-494.
22[Go31], p.9.
23[Go31].
24[Go31], Theorem VI, p.24.
25In his argument, Gödel refers to this formula only by its ‘Gödel’ number ‘$r$’: [Go31], p.25, Eqn.(12).
26[Go31], p.25(1) & p.26(2).
27[Go31], p.27.
28This follows from Gödel’s argument in [Go31], p.26(1).
29[Go31], p.22, Definitions #44-46.
Now:

**Lemma 12** In every sound interpretation of \( P \), the sequence \([F_1], \ldots, [F_n]\) of \( P \)-formulas interprets as a finite sequence \( F_1^*, \ldots, F_n^* \) of arithmetical propositions such that \( F_1^* \) is the interpretation of \( [(\forall x)R(x)] \), \( F_n^* \) is the interpretation of \( [\neg(\forall x)R(x)] \), and, for \( 2 \leq i \leq n \), \( F_i^* \) is either the interpretation of a \( P \)-axiom, or a logical consequence of the preceding formulas in the sequence by the interpretation of the rules of inference of \( P \).

We thus have that:

**Lemma 13** There is no sound interpretation of \( P \) under which \( [(\forall x)R(x)] \) interprets as true and \( [\neg(\forall x)R(x)] \) as false.

Since both \( [(\forall x)R(x)] \) and \( [\neg(\forall x)R(x)] \) are closed \( P \)-formulas, it further, follows that:

**Lemma 14** The formula \( [(\forall x)R(x) \rightarrow \neg(\forall x)R(x)] \) interprets as true under every sound interpretation of \( P \).

In other words, the implication ‘If \( F_1^* \) then \( F_n^* \)’ holds in any sound interpretation of \( P \). By Gödel’s completeness theorem, it follows that:

**Lemma 15** \( [(\forall x)R(x) \rightarrow \neg(\forall x)R(x)] \) is \( P \)-provable.

**Gödel’s Completeness Theorem:** In any first-order predicate calculus, the theorems are precisely the logically valid well-formed formulas (i.e. those that are true in every model of the calculus).

Since \( [(A \rightarrow \neg A) \rightarrow \neg A] \) is a theorem of first-order logic (see [Mc64], p.32, Ex.1), we have, by Modus Ponens, that:

**Lemma 16** \( [\neg(\forall x)R(x)] \) is \( P \)-provable.

Now, Gödel also showed that:

**Lemma 17** If \( P \) is consistent, then \( [R(n)] \) is \( P \)-provable for any given \( P \)-numeral \( n \).

It follows that:

**Theorem 3** \( P \) is not \( \omega \)-consistent.

Since Gödel’s argument holds in \( PA \), we further have that:

**Theorem 4** A consistent \( PA \) is not \( \omega \)-consistent.

**Corollary 2** Hilbert’s \( \omega \)-rule does not hold for a consistent \( PA \).

Thus Gödel’s Theorem VI of his 1931 paper is vacuously true, and does not establish the existence of a formally undecidable proposition in \( P \).

The above argument also applies to J. Barkley Rosser’s extension of Gödel’s reasoning since—as I show in a companion paper—Rosser’s argument implicitly appeals to Aristotle’s particularisation; thus, despite his claim of having assumed only simple consistency for \( P \), Rosser’s argument presumes that \( P \) is \( \omega \)-consistent.

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30 [Go31], p.26(2).
31 A formal argument is given in [An08a].
33 [Ro36].
34 [An09a].
3.4.1 The ‘standard’ interpretation of PA is not sound

It also follows that a sound interpretation of PA cannot appeal to Aristotle’s particularisation, and so:

Theorem 5 The interpretation $I_{PA}$(Standard/Tarski) of PA is not sound.$\square$

In other words—as Brouwer had noted$^{35}$—the phrase, ‘There is some $x$ such that $R^*(x)$’, is not always capable of an unambiguous meaning that can be represented in a formal language by the formula ‘$[(\exists x)R(x)]$’.

Moreover—as I show in a companion paper ($^{[An09b]}$)—we can define a finitary interpretation $I_{PA}$(Brouwer/Turing) of PA under which the PA-formula ‘$[(\exists x)R(x)]$’ does not interpret as ‘There is some $x$ such that $R^*(x)$ is true’.

Further, since the interpretation $I_{PA}$(Brouwer/Turing) is constructively sound, it can be shown that PA is categorical.

References


$^{35}$[Br08]


