

The significance of Aristotle's particularisation in the foundations of mathematics, logic and computability III

Rosser and formally undecidable arithmetical propositions

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Abstract

The logic underlying our current interpretations of all first-order formal languages—which provide the formal foundations for all computing languages—is Aristotle's logic of predicates. I review Rosser's claim that Gödel's reasoning can be recast to arrive at his intended result without the assumption of ω -consistency, since Rosser's argument appeals to a fundamental tenet of this logic, namely Aristotelean particularisation, which implies ω -consistency.¹

1 Introduction

Classical theory—following the 2000-year old philosophical perspective of Greek philosophers such as Aristotle—asks only that the 'core' axioms of a formal language, and its rules of inference, be interpretable—under Tarski's definitions of the satisfiability, and truth, of the formulas of a formal language under an interpretation—as self-evidently sound² propositions.

The subjective—and essentially irresolvable—element in the soundness of such a viewpoint is self-evident.

In these investigations I highlight—and show how to avoid—the limitations of such subjectivity in the case of the Peano Arithmetic, PA, by requiring that the 'core' axioms of PA, and its rules of inference, be interpretable as algorithmically (and, ipso facto, objectively) verifiable 'sound' propositions, and consider some consequences.

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²The word 'sound' is occasionally used informally as a synonym for 'reliable', but mostly as the technical term of mathematical logic defined precisely in the next section. The distinction is assumed to be obvious from the context in which it is used.

1.1 Preliminary Definitions and Comments

Aristotelean particularisation: This holds that an assertion such as, ‘There exists an x such that $F(x)$ holds’—usually denoted symbolically by ‘ $(\exists x)F(x)$ ’—can be validly inferred in the classical, Aristotelean, logic of predicates³ from the assertion, ‘It is not the case that: for any given x , $F(x)$ does not hold’—usually denoted symbolically by ‘ $\neg(\forall x)\neg F(x)$ ’.

Notation: I shall henceforth use square brackets to indicate that the contents represent a symbol or a formula of a formal theory, generally assumed to be well-formed unless otherwise indicated by the context.

In other words, expressions inside the square brackets are to be only viewed syntactically as juxtaposition of symbols that are to be formed and manipulated upon strictly in accordance with specific rules for such formation and manipulation—in the manner of a mechanical or electronic device—without any regards to what the symbolism might represent semantically under an interpretation that gives them meaning.

Moreover, even though the formula ‘ $[R(x)]$ ’ of a formal Arithmetic may interpret as the arithmetical relation expressed by ‘ $R^*(x)$ ’, the formula ‘ $[(\exists x)R(x)]$ ’ need not interpret as the arithmetical proposition denoted by the abbreviation ‘ $(\exists x)R^*(x)$.’ The latter denotes the phrase ‘There is some x such that $R^*(x)$ ’. As Brouwer had noted^{4,5}, this concept is not always capable of an unambiguous meaning that can be represented in a formal language by the formula ‘ $[(\exists x)R(x)]$ ’.

By ‘expressed’ I mean here that the symbolism is simply a short-hand abbreviation for referring to abstract concepts that may, or may not, be capable of a precise ‘meaning’. Amongst these are symbolic abbreviations which are intended to express the abstract concepts—particularly those of ‘existence’—involved in propositions that refer to non-terminating processes and infinite aggregates.

Provability: A formula $[F]$ of a formal system S is provable in S (S -provable) if, and only if, there is a finite sequence of S -formulas $[F_1], [F_2], \dots, [F_n]$ such that $[F_n]$ is $[F]$ and, for all $1 \leq i \leq n$, $[F_i]$ is either an axiom of S or a consequence of the axioms of S , and the formulas preceding it in the sequence, by means of the rules of deduction of S .

The structure \mathcal{N} : The structure of the natural numbers—namely, $\{N$ (*the set of natural numbers*); $=$ (*equality*); $'$ (*the successor function*); $+$ (*the addition function*); $*$ (*the product function*); 0 (*the null element*) $\}$.

The axioms of first-order Peano Arithmetic (PA)

PA₁: $[(x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))]$;

PA₂: $[(x_1 = x_2) \rightarrow (x'_1 = x'_2)]$;

PA₃: $[0 \neq x'_1]$;

PA₄: $[(x'_1 = x'_2) \rightarrow (x_1 = x_2)]$;

PA₅: $[(x_1 + 0) = x_1]$;

PA₆: $[(x_1 + x'_2) = (x_1 + x_2)']$;

PA₇: $[(x_1 * 0) = 0]$;

PA₈: $[(x_1 * x'_2) = ((x_1 * x_2) + x_1)]$;

PA₉: For any well-formed formula $[F(x)]$ of PA:

$[(F(0) \rightarrow (\forall x)(F(x) \rightarrow F(x')))] \rightarrow (\forall x)F(x)$.

³[HA28], pp.58-59.

⁴[Br08].

⁵See also [An08a].

Modus Ponens in PA: If $[A]$ and $[A \rightarrow B]$ are PA-provable, then so is $[B]$.

Generalisation in PA: If $[A]$ is PA-provable, then so is $[(\forall x)A]$.

Standard interpretation of PA: The standard interpretation $\mathcal{I}_{PA(Standard/Tarski)}$ of PA over the structure \mathcal{N} is the one in which the logical constants have their ‘usual’ interpretations⁶ in Aristotle’s logic of predicates, and⁷:

- (a) the set of non-negative integers is the domain;
- (b) the integer 0 is the interpretation of the symbol $[0]$;
- (c) the successor operation (addition of 1) is the interpretation of the $[']$ function;
- (d) ordinary addition and multiplication are the interpretations of $[+]$ and $[*]$;
- (e) the interpretation of the predicate letter $[=]$ is the identity relation.

Simple consistency: A formal system S is simply consistent if, and only if, there is no S-formula $[F(x)]$ for which both $[(\forall x)F(x)]$ and $[\neg(\forall x)F(x)]$ are S-provable.

ω -consistency: A formal system S is ω -consistent if, and only if, there is no S-formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S-provable and, second, $[F(a)]$ is S-provable for any given S-term $[a]$.

Soundness (formal system): A formal system S is sound under an interpretation \mathcal{I}_S if, and only if, every theorem $[T]$ of S translates as ‘ $[T]$ is true under \mathcal{I}_S ’.

Soundness (interpretation): An interpretation \mathcal{I}_S of a formal system S is sound if, and only if, S is sound under the interpretation \mathcal{I}_S .

Soundness in classical logic. In classical logic, a formal system S is defined as ‘sound’ if, and only if, it has an interpretation; and an interpretation is defined as the assignment of meanings to the symbols, and truth-values to the sentences, of the formal system. Moreover, any such interpretation is a model of the formal system. This definition suffers, however, from an implicit circularity: the formal logic L underlying any interpretation of S is implicitly assumed to be ‘sound’. The above definitions seek to avoid this implicit circularity by delinking the defined ‘soundness’ of a formal system under an interpretation from the implicit ‘soundness’ of the formal logic underlying the interpretation. This admits the case where, even if L_1 and L_2 are implicitly assumed to be sound, $S + L_1$ is sound, but $S + L_2$ is not. Moreover, an interpretation of S is now a model for S if, and only if, it is sound.⁸

Categoricity: A formal system S is categorical if, and only if, it has a sound interpretation and any two sound interpretations of S are isomorphic.⁹

2 Gödel and formally undecidable arithmetical propositions

In his seminal 1931 paper, Kurt Gödel showed that ([Go31], Theorem VI, p.24, p.25(1) & p.26(2)):

⁶See [Me64], p.49.

⁷See [Me64], p.107.

⁸My thanks to Professor Rohit Parikh for highlighting the need for making such a distinction explicit.

⁹Compare [Me64], p.91.

Lemma 1 : *If a Peano Arithmetic P is ω -consistent, then there is a constructively definable P -formula $[R(x)]$ ¹⁰ such that neither $[(\forall x)R(x)]$ nor $[\neg(\forall x)R(x)]$ are P -provable.*

Of course, since every ω -consistent system is necessarily simply consistent, Gödel's conclusion is significant only if there is an ω -consistent language that seeks to formally express all our true propositions about the natural numbers.

The issue, of whether there is an ω -consistent system of Arithmetic at all, appears to have been treated as inconsequential¹¹ following J. Barkley Rosser's 1936 paper ([Ro36]), in which he claimed that Gödel's reasoning can be recast to arrive at Gödel's intended result (i.e., construction of a formally undecidable arithmetical proposition in P) by assuming only that P is simply consistent (i.e., without assuming that P is ω -consistent).

In this paper, I analyse Rosser's argument, and show, first, that it implicitly appeals to Aristotle's particularisation; and, second, that such appeal is valid only if P is ω -consistent.

2.1 The significance of ω -consistency

Now, under Tarski's definitions of the satisfaction, and truth, of the formulas of a first-order language under an interpretation¹², the formally defined logical constant ' $[\exists]$ ' in an occurrence such as ' $[(\exists x) \dots]$ '—which is formally defined in terms of the primitive (undefined) logical constant ' $[\forall]$ ' as ' $[\neg(\forall x)\neg \dots]$ '—*always* appeals to an interpretation such as 'There is some x such that \dots ' in any sound interpretation of *any* formal first-order mathematical language¹³.

In other words:

Lemma 2 : *If the first-order predicate calculus of a first-order mathematical language admits quantification, then any putative model of the language must interpret existential quantification as Aristotle's particularisation under Tarski's definitions of the satisfaction, and truth, of the formulas of a first-order language under an interpretation.*

We thus have:

Lemma 3 : *If Aristotle's particularisation is logically valid, then the standard interpretation $\mathcal{I}_{PA(Standard/Tarski)}$ of PA is sound (under Tarski's definitions).*

Lemma 4 : *If $\mathcal{I}_{PA(Standard/Tarski)}$ is sound, then PA is ω -consistent.*

3 Rosser and formally undecidable arithmetical propositions

Although both Gödel's proof and Rosser's argument are complex, and not easy to unravel, the former has been extensively analysed, and its various steps for-

¹⁰In his argument, Gödel refers to this formula only by its 'Gödel' number ' r '; [Go31], p.25, Eqn.(12).

¹¹See, for instance, [Be59], p.595; [Wa63], p.19 (Theorem 3) & p.25; [Me64], p.144; [Sh67], p.132 (Incompleteness Theorem); [EC89], p.215; [BBJ03], p.224 (Gödel's first incompleteness theorem).

¹²[Ta33].

¹³See, for instance, [Me64], p.52(ii).

mally validated¹⁴, in a number of expositions of Gödel’s number-theoretic reasoning¹⁵.

3.1 Expositions of Rosser’s argument are generally sketchy

In sharp contrast, Rosser’s widely cited¹⁶ argument does not appear to have received the same critical scrutiny, and its number-theoretic expositions generally remain either implicit or sketchy¹⁷.

3.1.1 Wang’s outline of Rosser’s argument

Wang, for instance, states that¹⁸ from the formal provability of:

$$(i) \neg(x)(B(x, \bar{q}) \supset (Ey)(y \leq x \ \& \ B(y, n(\bar{q}))))$$

in his formal system of first-order Peano Arithmetic Z, we may infer the formal provability of:

$$(ii) (Ex)(B(x, \bar{q}) \ \& \ \neg(Ey)(y \leq x \ \& \ B(y, n(\bar{q}))))$$

However, the inference (ii) from (i) appears to assume that the following deduction is valid for some \bar{j} :

$$\begin{aligned} &\neg(x)(B(x, \bar{q}) \supset (Ey)(y \leq x \ \& \ B(y, n(\bar{q})))) \\ &(Ex)\neg(B(x, \bar{q}) \supset (Ey)(y \leq x \ \& \ B(y, n(\bar{q})))) \\ &\neg(B(\bar{j}, \bar{q}) \supset (Ey)(y \leq \bar{j} \ \& \ B(y, n(\bar{q})))) \\ &B(\bar{j}, \bar{q}) \ \& \ \neg(Ey)(y \leq \bar{j} \ \& \ B(y, n(\bar{q}))) \\ &(Ex)(B(x, \bar{q}) \ \& \ \neg(Ey)(y \leq x \ \& \ B(y, n(\bar{q})))) \end{aligned}$$

Thus, Wang’s conclusion appears to implicitly assume that the ‘standard’ interpretation of PA, when applied to his Peano Arithmetic Z, is sound; and, ipso facto, that Z is ω -consistent.

Although Wang does not explicitly define the interpretation of the formal Z-formula ‘ $(Ex)F(x)$ ’ as ‘There is some x such that $F(x)$ ’, this interpretation appears implicit in his discussion and definition of ‘ $(Ev)A(v)$ ’ in terms of Hilbert’s ε -function¹⁹ as a property of the underlying logic of Wang’s Peano Arithmetic Z, and is obvious in the above argument.

In other words Wang implicitly implies that the interpretation of existential quantification cannot be specific to any particular interpretation of a formal mathematical language, but must necessarily be determined by the predicate calculus that is to be applied uniformly to all the mathematical languages in question.

¹⁴Possibly because Gödel’s remarkably self-contained 1931 paper—it neither contained, nor needed, any formal citations—remains unsurpassed in mathematical literature for thoroughness, clarity, transparency and soundness of exposition.

¹⁵For instance [Me64], p.143; [EC89], p.210-211.

¹⁶[Be59], pp.393-395 (which focuses on Rosser’s argument, and treats Gödel’s proof of his Theorem VI ([Go31], p.24) as a, secondary, weaker result); [Wa63], p.337; [Me64], pp.144-147; [Sh67], p.232 (interestingly, this introductory text contains *no* reference to Gödel or to his 1931 paper!); [EC89], p.215; [Sm92], p.81; [BBJ03], p.226 (this introductory text, too, focuses on Rosser’s argument, and treats Gödel’s argument as more of a historical curiosity!).

¹⁷See [Be59], p.593-595; [Wa63], p.337; [Sh67], p.232; [Rg87], p.98; [EC89], p.217, Ex.2; [Sm92], p.81; [BBJ03], p.226.

¹⁸[Wa63], p.337.

¹⁹[Wa63], p.315(2.31); see also p.10 & pp.443-445.

3.1.2 Beth's outline of Rosser's argument

Similarly, in his outline of a formalisation of Rosser's argument, Beth implicitly concludes²⁰ that from the formal provability of:

$$(i) \neg(q)[G_1(m^0, q, m^0) \rightarrow (s)\{B(s, q) \rightarrow (Et)[t \leq s \ \& \ (Er)\{H(q, r) \ \& \ B(t, r)\}]\}]$$

in his formal system of first-order Peano Arithmetic P, we may infer the formal provability of:

$$(ii) (Eq)[G_1(m^0, q, m^0) \ \& \ (s)\{B(s, q) \ \& \ (t)[t \leq s \rightarrow (r)\{H(q, r) \rightarrow \overline{B(t, r)}\}]\}]$$

However, again, the inference (ii) from (i) appears to assume that the following deduction is valid for some \bar{j} :

$$\begin{aligned} & \neg(q)[G_1(m^0, q, m^0) \rightarrow (s)\{B(s, q) \rightarrow (Et)[t \leq s \ \& \ (Er)\{H(q, r) \ \& \ B(t, r)\}]\}] \\ & (Eq)\neg[G_1(m^0, q, m^0) \rightarrow (s)\{B(s, q) \rightarrow (Et)[t \leq s \ \& \ (Er)\{H(q, r) \ \& \ B(t, r)\}]\}] \\ & \neg[G_1(m^0, \bar{j}, m^0) \rightarrow (s)\{B(s, \bar{j}) \rightarrow (Et)[t \leq s \ \& \ (Er)\{H(\bar{j}, r) \ \& \ B(t, r)\}]\}] \\ & G_1(m^0, \bar{j}, m^0) \ \& \ (s)\{B(s, \bar{j}) \ \& \ (t)[t \leq s \rightarrow (r)\{H(\bar{j}, r) \rightarrow \overline{B(t, r)}\}]\}] \\ & (Eq)[G_1(m^0, q, m^0) \ \& \ (s)\{B(s, q) \ \& \ (t)[t \leq s \rightarrow (r)\{H(q, r) \rightarrow \overline{B(t, r)}\}]\}] \end{aligned}$$

Thus, Beth's conclusion, too, appears to implicitly assume that the 'standard' interpretation of PA, when applied to Beth's Peano Arithmetic P, is sound; and, ipso facto, that P is ω -consistent.

In this case, Beth explicitly defines the interpretation of the formal P-formula ' (Ex) ' as 'There is a value of x such that'²¹.

Thus Beth, too, implies that the interpretation of existential quantification in formalised axiomatics cannot be specific to any particular interpretation of a formal mathematical language, but must necessarily be determined by the predicate calculus that is to be applied uniformly to all the mathematical languages in question.

4 Rosser's argument presumes ω -consistency

Now, Rosser's claim in his 'extension'²² of Gödel's argument²³ is that, whereas Gödel's argument assumes that his Peano Arithmetic, P, is ω -consistent, Rosser's assumes only simple consistency.

However, Rosser's original argument (also a sketch) appears to presume that the system of Peano Arithmetic in question is ω -consistent.

For instance, the concluding deduction in Rosser's reasoning presumes that if, for any given natural number n , the formula in Gödel's Peano Arithmetic P whose Gödel-number is:

$$Neg(Sb(r \overset{u}{Z(n)} \overset{v}{Z(a)}))$$

is P_κ -provable²⁴ under the given premises, we may conclude that, if P is simply consistent, then the P-formula whose Gödel-number is:

²⁰[Be59], p.594 (ij).

²¹[Be59], p.178.

²²[Ro36].

²³[Go31].

²⁴Notation (due to Gödel): By ' P_κ -provable' we mean provable from the axioms of P and an arbitrary class, κ , of P-formulas—including the case where κ is empty—by the rules of deduction of P.

$$uGen(Neg(Sb(r \overset{v}{Z}(a))))$$

is also P_κ -provable.

Rosser essentially seems to reason here that since the P-formula $[\neg R(n, a)]$ —where $[a]$ is a specific P-numeral—is P_κ -provable for any given P-numeral $[n]$, we may conclude that the P-formula $[(\forall u)\neg R(u, a)]$ is P_κ -provable. This would presume, however, that P is ω -consistent.

4.1 Where Rosser’s argument presumes ω -consistency

I consider a more formal expression²⁵ of Rosser’s argument and highlight where it implicitly presumes that P is ω -consistent.

Now, Gödel²⁶ defines a primitive recursive relation, $q(x, y)$, that holds if, and only if, x is the Gödel-number of a well-formed P-formula²⁷, say $[H(w)]$ —which has a single free variable, $[w]$ —and y is the Gödel-number of a P-proof of $[H(x)]$. So, for any natural numbers h, j :

(a) $q(h, j)$ holds if, and only if, j is the Gödel-number of a P-proof of $[H(h)]$.

Rosser’s argument defines an additional primitive recursive relation, $s(x, y)$, which holds if, and only if, x is the Gödel-number of $[H(w)]$, and y is the Gödel-number of a P-proof of $[\neg H(x)]$. Hence, for any natural numbers h, j :

(b) $s(h, j)$ holds if, and only if, j is the Gödel-number of a P-proof of $[\neg H(h)]$.

Further, it follows from Gödel’s Theorems V²⁸ and VII²⁹ that the primitive recursive relations $q(x, y)$ and $s(x, y)$ are instantiationally equivalent to some arithmetical relations, $Q(x, y)$ and $S(x, y)$, such that, for any natural numbers h, j :

- (c) If $q(h, j)$ holds, then $[Q(h, j)]$ is P-provable;
- (d) If $\neg q(h, j)$ holds, then $[\neg Q(h, j)]$ is P-provable;
- (e) If $s(h, j)$ holds, then $[S(h, j)]$ is P-provable;
- (f) If $\neg s(h, j)$ holds, then $[\neg S(h, j)]$ is P-provable;

Now, whilst Gödel defines $[H(w)]$ as $[(\forall y)\neg Q(w, y)]$, Rosser’s argument defines $[H(w)]$ as $[(\forall y)(Q(w, y) \rightarrow (\exists z)(z \leq y \wedge S(w, z)))]$,

Further, whereas Gödel considers the P-provability of the Gödelian proposition, $[(\forall y)\neg Q(h, y)]$, Rosser’s argument considers the P-provability of the proposition $[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]$.

We note that, by definition:

(i) $q(h, j)$ holds if, and only if, j is the Gödel-number of a P-proof of:

$$[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))];$$

(ii) $s(h, j)$ holds if, and only if, j is the Gödel-number of a P-proof of:

$$[\neg((\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]).$$

²⁵eg. [Me64], p.145, Proposition 3.32.

²⁶[Go31], p.24, 8.1.

²⁷Of his formally defined Peano Arithmetic, P.

²⁸[Go31], p.22.

²⁹[Go31], p.29.

4.2 The formal expression of Rosser's argument

(a) We assume, first, that r is the Gödel-number of some proof sequence in P for the proposition $[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]$.

Hence $q(h, r)$ is true, and $[Q(h, r)]$ is P -provable.

However, we then have that $[Q(h, r) \rightarrow (\exists z)(z \leq r \wedge S(h, z))]$ is P -provable.

Further, by Modus Ponens, we have that $[(\exists z)(z \leq r \wedge S(h, z))]$ is P -provable.

Now, if P is simply consistent, then $[\neg((\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z))))]$ is not P -provable.

Hence, $s(h, n)$ does not hold for any natural number n , and so $\neg s(h, n)$ holds for every natural number n .

It follows that $[\neg S(h, n)]$ is P -provable for every P -numeral $[n]$.

Hence, $[\neg((\exists z)(z \leq r \wedge S(h, z)))]$ is also P -provable - a contradiction.

Hence, $[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]$ is not P -provable if P is simply consistent.

(b) We assume next that r is the Gödel-number of some proof-sequence in P for the proposition $[\neg((\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z))))]$.

Hence $s(h, r)$ holds, and $[S(h, r)]$ is P -provable.

However, if P is simply consistent, $[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]$ is not P -provable.

Hence, $\neg q(h, n)$ holds for every natural number n , and $[\neg Q(h, n)]$ is P -provable for all P -numerals $[n]$.

(i) The foregoing implies $[y \leq r \rightarrow \neg Q(h, y)]$ is P -provable, and we consider the following deduction³⁰:

- | | |
|--|--------------------------|
| (1) $[r \leq k]$ | ... <i>Hypothesis</i> |
| (2) $[S(h, r)]$ | ... <i>By 3(b)</i> |
| (3) $[r \leq k \wedge S(h, r)]$ | ... <i>From (1), (2)</i> |
| (4) $[(\exists z)(z \leq k \wedge S(h, z))]$ | ... <i>From (3)</i> |

(ii) From (1)-(4), by the Deduction Theorem, we have that $[r \leq k \rightarrow (\exists z)(z \leq k \wedge S(h, z))]$ is provable in P for any P -numeral $[k]$;

(iii) Now, $[k \leq r \vee r \leq k]$ is P -provable for any P -numeral $[k]$;

(iv) Also, $[(k \leq r \rightarrow \neg Q(h, k)) \wedge (r \leq k \rightarrow (\exists z)(z \leq k \wedge S(h, z)))]$ is P -provable for any P -numeral $[k]$.

(v) Hence $[(\neg(k \leq r) \vee \neg Q(h, k)) \wedge (\neg(r \leq k) \vee (\exists z)(z \leq k \wedge S(h, z)))]$ is P -provable for any P -numeral $[k]$.

(vi) Hence $[\neg Q(h, k) \vee (\exists z)(z \leq k \wedge S(h, z))]$ is P -provable for any P -numeral $[k]$.

(vii) Hence $[(Q(h, k) \rightarrow (\exists z)(z \leq k \wedge S(h, z)))]$ is P -provable for any P -numeral $[k]$.

(viii) Now, (vii) contradicts our assumption that $[\neg((\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z))))]$ is P -provable.

³⁰cf. [Me64], p.146.

(ix) Hence $[\neg((\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z))))]$ is not P-provable if P is simply consistent.

However, the claimed contradiction in (viii) *only* follows if we assume that P is ω -consistent, and *not* if we assume only that P is simply consistent.

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