Abstract

We define a finitary model of first-order Peano Arithmetic in which satisfaction and quantification are interpreted constructively in terms of Turing-computability.

1 Introduction

There is a remarkable, albeit unremarked, consequence of Turing’s seminal 1936 paper ([Tu36], pp. 230–265).

It admits a constructive definition of what is meant by the assertion that the axioms/rules of inference of a first-order Peano Arithmetic, PA, are satisfied/preserve truth under an interpretation.

1.1 The theorems of PA are Turing-computable

Specifically, let \([N]\) denote the usual structure—defined as \(\{N (the set of natural numbers); = (equality); ' (the successor function); + (the addition function); * (the product function); 0 (the null element)\}\)—that serves for a definition of today’s standard interpretation (cf. [Mc64], section §2, pp.49-53; p107), say \(M\), of PA.

Meta-theorem: If a formula \([R]\) is a theorem of a first-order Peano Arithmetic then there is a Turing-machine \(TM_R\) that will compute the arithmetical proposition—or relation—\(R\) as true—or always true (i.e., true for any natural number values assigned to the variables of \(R\) ), respectively—in a finite number of steps.
Proof: Consider the PA-axioms:

- **A1:** \((x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))\);
- **A2:** \((x_1 = x_2) \rightarrow (x_1' = x_2')\);
- **A3:** \(0 \neq x_1'\);
- **A4:** \((x_1' = x_2') \rightarrow (x_1 = x_2)\);
- **A5:** \((x_1 + 0) = x_1\);
- **A6:** \((x_1 + x_2') = (x_1 + x_2)'\);
- **A7:** \((x_1 \cdot 0) = 0\);
- **A8:** \((x_1 \cdot x_2') = ((x_1 \cdot x_2) + x_1)\);
- **A9:** For any well-formed formula \(F(x)\) of PA:
  
  \[
  ([F(0) \rightarrow \forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x).\]

We define the following, Turing-computable, interpretation \(\beta\) of PA over the structure \([N]\), and show that \(\beta\) is a finitary model of PA, since all PA-provable formulas can be shown to interpret constructively as arithmetical propositions/relations that are Turing-computable as true/always true over \([N]\) in the following, definitional, sense (compare section §2, pp.49-53):

**D1:** If a total number-theoretical relation, \(R(x_1, x_2, \ldots, x_n)\), when treated as a Boolean function, defines the Turing-machine \(T_M\), then \(R(x_1, x_2, \ldots, x_n)\) is \(T_M\)-computable if, and only if, for any given natural number sequence \((a_1, a_2, \ldots, a_n)\), \(T_M\) will compute \(R(a_1, a_2, \ldots, a_n)\) as either true, or as false, over \([N]\).

**D2:** If \([R]\) is an atomic formula \([R(a_1, a_2, \ldots, a_n)]\) of PA, then the natural number sequence \((a_1, a_2, \ldots, a_n)\) satisfies \([R]\) under the interpretation \(\beta\) if, and only if, the arithmetical relation \(R(x_1, x_2, \ldots, x_n)\) is \(T_M\)-computable as true over \([N]\) on the natural number input \((a_1, a_2, \ldots, a_n)\).

**D3:** The natural number sequence \((a_1, a_2, \ldots, a_n)\) satisfies the PA-formula \([-R]\) under the interpretation \(\beta\) if, and only if, \((a_1, a_2, \ldots, a_n)\) does not satisfy the PA-formula \([R]\) under \(\beta\).

**D4:** The natural number sequence \((a_1, a_2, \ldots, a_n)\) satisfies the PA-formula \(([R \rightarrow S]\) under the interpretation \(\beta\) if, and only if, either \((a_1, a_2, \ldots, a_n)\) does not satisfy the PA-formula \([R]\) or \((a_1, a_2, \ldots, a_n)\) satisfies the PA-formula \([S]\) under \(\beta\).

**D5:** The natural number sequence \((a_1, a_2, \ldots, a_n)\) satisfies the PA-formula \(\forall x_1)R\) under the interpretation \(\beta\) if, and only if, every sequence \((b_1, b_2, \ldots, b_n)\) that differs from \((a_1, a_2, \ldots, a_n)\) in at most the \(i^{th}\) component satisfies \([R]\) under \(\beta\).

**D6:** The PA-formula \([R]\) is true under the interpretation \(\beta\) if, and only if, every natural number sequence \((a_1, a_2, \ldots, a_n)\) satisfies \([R]\) under \(\beta\).

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1. Wherever we use the word “all” or “every”, it is to be read as synonymous with the phrase “any given”.
2. In the sense of mechanically, without the use of any human ingenuity.
3. We take (Mc64, section §2, pp.49-53) as representative of a standard exposition of Tarski’s definitions of the satisfaction and truth of the formulas of a formal language under an interpretation.
4. It follows from Turing’s seminal 1936 paper on computable numbers that every quantifier-free arithmetical function (or relation, when interpreted as a Boolean function) \(F\) defines a Turing-machine \(T_M\) (cf. Tu36, pp. 138-139). In the general case, \(T_M\) is defined by the quantifier-free expression in the prenex normal form of \(F\).
D7: The PA-formula $\lnot R$ is true under the interpretation $\beta$ if, and only if, it is not the case that every natural number sequence $\langle a_1, a_2, \ldots, a_n \rangle$ satisfies $[R]$ under $\beta$.

Now if, for instance, the axiom $A1$ is true under the interpretation $\beta$, then it follows that $A1$ interprets in $\beta$ as an arithmetical relation that is TM$_{A1}$-computable as always true over $[N]$.

Similar arguments hold for the axioms $A2$ to $A8$.

Next, if the Induction axiom, $A9$, is true under the interpretation $\beta$, then it follows that $A9$ interprets in $\beta$ as the arithmetical proposition expressed by:

\[(F(0) \rightarrow (\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)\]

which is TM$_{A9}$-computable as true over $[N]$, since:

(a) If $[F(0)]$ is true under $\beta$, then $F(x)$ is TM$_F$-computable as true over $[N]$ on the input 0;

(b) If $[(\forall x)(F(x) \rightarrow F(x'))]$, say $[(\forall x)G]$, is true under $\beta$, then $G$ is TM$_G$-computable as always true over $[N]$;

(c) If $F(x)$ is TM$_F$-computable as true over $[N]$ on the input 0, and $(F(x) \rightarrow F(x'))$ is TM$_G$-computable as always true over $[N]$, then $F(x)$ is TM$_F$-computable as always true over $[N]$.

Further, the following rules of Inference in PA preserve truth under $\beta$:

- **Modus Ponens**: $[B]$ follows from $[A]$ and $[A \rightarrow B]$;
- **Generalisation**: $[(\forall x)A]$ follows from $[A]$.

In other words, whenever the PA-formulas $[A]$ and $[(A \rightarrow B)]$ interpret as true under $\beta$, the PA-formula $[B]$ also interprets as true under $\beta$ since:

(d) If $[A]$ interprets as true under $\beta$, then every natural number sequence $\langle a_1, a_2, \ldots, a_n \rangle$ satisfies $[A]$ under $\beta$;

(e) If $[(A \rightarrow B)]$ interprets as true under $\beta$, then, for any natural number sequence $\langle a_1, a_2, \ldots, a_n \rangle$, either $\langle a_1, a_2, \ldots, a_n \rangle$ does not satisfy the PA-formula $[A]$ or $\langle a_1, a_2, \ldots, a_n \rangle$ satisfies the PA-formula $[B]$ under $\beta$;

(f) If both $[A]$ and $[(A \rightarrow B)]$ interpret as true under $\beta$, then $[B]$ interprets as true under $\beta$, as any sequence $\langle a_1, a_2, \ldots, a_n \rangle$ must satisfy $[B]$ under $\beta$.

Similarly if $[A]$ interprets as true under $\beta$, then $[(\forall x)A]$ interprets as true under $\beta$ since:

(g) If $[A]$ interprets as true under $\beta$, then every natural number sequence $\langle a_1, a_2, \ldots, a_n \rangle$ satisfies $[A]$ under $\beta$;

(h) If $[(\forall x)A]$ interprets as true under $\beta$, then every natural number sequence $\langle a_1, a_2, \ldots, a_n \rangle$ satisfies $[A]$ under $\beta$.

Thus the axioms of PA are constructively satisfied/true$^7$ under the finitary interpretation $\beta$, and the rules of inference of PA preserve the properties of satisfaction/true$^8$ under $\beta$.

It follows that, if a formula $[R]$ is a theorem of PA, then there is a Turing-machine, TM$_R$ that will compute the arithmetical proposition—or relation—$R$.

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$^7$In an intuitionistically unobjectionable sense.

$^8$In an intuitionistically unobjectionable sense.
as true—or always true, respectively—in a finite number of steps. \(\Box\)

1.2 The finitary interpretation \(\beta\) of PA over \([N]\) is a model of PA

It follows from the above that the finitary interpretation \(\beta\) of PA over \([N]\) is constructively sound\(^9\) and so it defines a finitary model of PA.

2 The standard interpretation of PA over \([N]\)

The above, finitary, interpretation \(\beta\) of PA over \([N]\) differs from the standard interpretation \(M\) of PA over \([N]\) (cf. \[Me64\], section §2, pp.49-53; p107) in two respects only; first in the definition \(D2\), and second in the interpretation of existential quantification under \(M\).

Thus, first, the standard interpretation \(M\) replaces \(D2\) with the following, Platonic, definition—due to Tarski \[Ta33\]—of the satisfaction of the formula of a formal language such as PA under an interpretation such as \(M\) (cf. \[Me64\], p51(i)):

\[D2': \text{If } [R] \text{ is an atomic formula } [R(a_1,a_2,\ldots,a_n)] \text{ of PA, then the natural number sequence } (a_1,a_2,\ldots,a_n) \text{ satisfies } [R] \text{ under the interpretation } M \text{ if, and only if, } R(a_1,a_2,\ldots,a_n) \text{ holds over } [N].\]

(The above is, essentially, a formal expression of Tarski’s well-known definition: The sentence “Snow is white” is true if, and only if, snow is white.)

What makes \(D2'\) ‘Platonic’ is that—in striking contrast to \(D2\)—it gives no indication as to how we may constructively verify whether, or not, ‘the arithmetical relation \(R(a_1,a_2,\ldots,a_n)\) holds over \([N]\).’

Second, \(M\) also interprets the existential quantifier Platonically—essentially as proposed by Hilbert in terms of his \(\varepsilon\)-function (cf. \[An08a\])—as follows (cf. \[Me64\], p52 V(ii)):

\[D5': \text{The natural number sequence } (a_1,a_2,\ldots,a_n) \text{ satisfies the PA-formula } [(\exists x)R(x)] \text{ under the standard interpretation } M \text{ over } [N] \text{ if, and only if, there is a sequence } (b_1,b_2,\ldots,b_n) \text{ that differs from } (a_1,a_2,\ldots,a_n) \text{ in at most the } i^{th} \text{ component and } (b_1,b_2,\ldots,b_n) \text{ satisfies } [R].\]

Now, definition \(D5'\) essentially states that the PA-formula \([(\exists x)R(x)]\)—which is simply an abbreviation for the PA-formula \([\neg(\forall x)\neg R(x)]\)—is true under \(M\) if, and only if, it is not only the case that \(R(x)\) does not always hold in the domain of the interpretation, but we also have that \(R(a)\) holds for some \(a\) in the domain of the interpretation\(^{10}\).

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\(^9\)An interpretation of PA is sound if all PA-theorems are true under the interpretation.

\(^{10}\)This is the interpretation of the existential quantifier proposed by Hilbert in terms of his \(\varepsilon\)-function that was—justifiably (cf. \[An08a\])—objected to by Brouwer as non-constructive, and intuitionistically objectionable, on the grounds that it gives no indication as to how we may constructively verify that \(R(a)\) does, indeed, hold over \([N]\) for some \(a\).
2.1 The standard interpretation of PA over \([N]\) is not sound

However, it is shown in [An08a] that if \(D_5'\) is sound, then PA is necessarily \(\omega\)-consistent.

Now, PA is \(\omega\)-consistent if, and only if, there is no PA-formula such as \([F(x)]\) for which:

(i) \([\neg(\forall x)F(x)]\) is PA-provable, and,
(ii) \([F(n)]\) is PA-provable for any given numeral \([n]\) of PA.

Since it is also shown in [An08a] that PA is not \(\omega\)-consistent, it follows that the standard interpretation \(M\) of PA over \([N]\) is not sound.

3 The finitary interpretation \(\beta\) of PA over \([N]\) is consistent with an \(\omega\)-inconsistent PA

In sharp contrast, the remarkable feature of the finitary interpretation \(\beta\) is that if \('[\exists x]R(x)]'—formally defined as \('[\neg(\forall x)\neg R(x)]'—is PA-provable, we cannot conclude that there must exist a natural number \(n\) such that the arithmetical proposition \(R(n)\) is true over \([N]\).

We can only conclude constructively that \(R(n)\) is not computable by any Turing-machine \(TM_R\) as always false. However, \(R(n)\) may still be constructively computable as false for any natural number \(n\)!

References


\(^{11}\text{cf. [An08a], §1.1 First Tautology Theorem, §1.2 Second Tautology Theorem}\)}
International Conference on Foundations of Computer Science, July 14-17, 2008, Las Vegas, USA.