Rosser's proof of undecidability *implicitly* assumes ω-consistency

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1 INTRODUCTION

Rosser's claim in his 'extension' [Ro36] of Gödel's argument [Go31] is that, whereas Gödel's argument assumes that first-order Peano Arithmetic, PA, is ω-consistent, his assumes only simple consistency.

PA is defined as ω-consistent if, and only if, there is no PA-formula, say $[R(x)]$, for which:

(i) $[\neg(\forall x)R(x)]$ is PA-provable, and, also:

(ii) $[R(n)]$ is PA-provable for any given numeral $[n]$ of PA.

However, a formal expression (eg. [Me64]) of Rosser's argument as below shows that it also presumes PA is ω-consistent.

2 THE SIGNIFICANCE OF ω-CONSISTENCY

The standard model of PA *presumes*, following Hilbert (cf. [An08]), that the standard interpretation, say $\boldsymbol{M}$, under which the PA-formula $[(\exists x)R(x)]$—which is merely an abbreviation for $[\neg(\forall x)R(x)]$—interprets as true if, and only if, $R(n)$ holds for some natural number n under $\boldsymbol{M}$, is *sound*.

Clearly, such an interpretation of the existential quantifier immediately implies that PA is necessarily ω-consistent.

Since Brouwer's main objection (cf. [An08]) was to Hilbert's presumption that this particular interpretation of the existential quantifier is sound, Gödel explicitly avoided this assumption in his seminal 1931 paper ([Go31], p9) in order to ensure that his reasoning was acceptable as "constructive" and "intuitionistically unobjectionable" ([Go31], p26).

He chose, instead, to present the formal undecidability of his arithmetical proposition—and the consequences arising from it—as explicitly conditional on the assumption of the weaker property of ω-consistency for PA, under the belief that:

> Every ω-consistent is obviously also consistent ([Go31], p24).

However, in [An08] we show that Gödel's belief was misplaced, and that Brouwer's objection to Hilbert's presumption—that the above interpretation of the existential quantifier is sound—was justified since, if PA is consistent, then it is provably ω-*in*consistent; from which it follows that the standard interpretation $\boldsymbol{M}$ of PA is *not* sound.

3 ROSSER'S PROPOSITION

Now, Gödel ([Go31], p24, 8.1) defines a primitive recursive relation, $q(x, y)$, that holds if, and only if, x is the Gödel-number of a well-formed PA-formula, say $[H(w)]$—which has a single free variable, $[w]$—and y is the Gödel-number of a PA-proof of $[H(x)]$.

So, for any natural numbers h, j:

(a) $q(h, j)$ holds if, and only if, j is the Gödel-number of a PA-proof of $[H(h)]$.

Rosser's argument defines an additional primitive recursive relation, $s(x, y)$, which holds if, and only if, x is the Gödel-number of $[H(w)]$, and y is the Gödel-number of a PA-proof of $[\neg H(x)]$.

Hence, for any natural numbers h, j:

(b) $s(h, j)$ holds if, and only if, j is the Gödel-number of a PA-proof of $[\neg H(h)]$.

Further, by Gödel's Theorems V ([Go31], p22) and VII ([Go31], p29), the primitive recursive relations $q(x, y)$ and $s(x, y)$ are instantiationally equivalent to some arithmetical relations, $Q(x, y)$ and $S(x, y)$, such that, for any natural numbers h, j:

(c) If $q(h, j)$ holds, then $[Q(h, j)]$ is PA-provable;
(d) If $\neg q(h, j)$ holds, then $[\neg Q(h, j)]$ is PA-provable;
(e) If $s(h, j)$ holds, then $[S(h, j)]$ is PA-provable;
(f) If $\neg s(h, j)$ holds, then $[\neg S(h, j)]$ is PA-provable;

Now, whilst Gödel defines $[H(w)]$ as $[(\forall y)\neg Q(w, y)]$, Rosser's argument defines $[H(w)]$ as $[(\forall y)(Q(w, y) \rightarrow (\exists z)(z \leq y \wedge S(w, z)))]$,

Further, whereas Gödel considers the PA-provability of the Gödelian proposition, $[(\forall y)\neg Q(h, y)]$, Rosser's argument considers the PA-provability of the proposition $[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]$.

We note that, by definition:

(i) $q(h, j)$ holds if, and only if, j is the Gödel-number of a PA-proof of:

$[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]$;

(ii) $s(h, j)$ holds if, and only if, j is the Gödel-number of a PA-proof of:

$[\neg((\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z))))]$.

4 ROSSER'S ARGUMENT

(a) We assume, first, that r is the Gödel-number of some proof sequence in PA for the proposition $[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]$.

Hence $q(h, r)$ is true, and $[Q(h, r)]$ is PA-provable.

However, we then have that $[Q(h, r) \rightarrow (\exists z)(z \leq r \wedge S(h, z))]$ is PA-provable.

Further, by Modus Ponens, we have that $[(\exists z)(z \leq r \wedge S(h, z)))]$ is PA-provable.

Now, if PA is simply consistent, then $[\neg((\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z))))]$ is not PA-provable.

Hence, $s(h, n)$ does not hold for any natural number n, and so $\neg s(h, n)$ holds for every natural number n.

It follows that $[\neg S(h, n)]$ is PA-provable for every PA-numeral $[n]$.

Hence, $[\neg((\exists z)(z \leq r \wedge S(h,z)))]$ is also PA-provable - a contradiction.

Hence, $[(\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z)))]$ is not PA-provable if PA is simply consistent.

(b) We assume next that r is the Gödel-number of some proof-sequence in PA for the proposition $[\neg((\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z))))]$.

Hence $s(h,r)$ holds, and $[S(h,r)]$ is PA-provable.

However, if PA is simply consistent, $[(\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z)))]$ is not PA-provable.

Hence, $\neg q(h,n)$ holds for every natural number n, and $[\neg Q(h,n)]$ is PA-provable for all PA-numerals $[n]$.

(i) The foregoing implies $[y \leq r \rightarrow \neg Q(h,y)]$ is PA-provable, and we consider the following deduction (cf. 1964. Mendelson, p146):

(1) $[r \leq k]$	... *Hypothesis*
(2) $[S(h,r)]$	... *By 3(b)*
(3) $[r \leq k \wedge S(h,r)]$	... *From (1) and (2); Tautology*
(4) $[(\exists z)(z \leq k \wedge S(h,z))]$	... *From (3); logical axioms of PA*

(ii) From (1)-(4), by the Deduction Theorem, we have that $[r \leq k \rightarrow (\exists z)(z \leq k \wedge S(h,z))]$ is provable in PA for any PA-numeral $[k]$;

(iii) Now, $[k \leq r \vee r \leq k]$ is PA-provable for any PA-numeral $[k]$;

(iv) Also, $[(k \leq r \rightarrow \neg Q(h,k)) \wedge (r \leq k \rightarrow (\exists z)(z \leq k \wedge S(h,z)))]$ is PA-provable for any PA-numeral $[k]$.

(v) Hence $[(\neg(k \leq r) \vee \neg Q(h,k)) \wedge (\neg(r \leq k) \vee (\exists z)(z \leq k \wedge S(h,z)))]$ is PA-provable for any PA-numeral $[k]$.

(vi) Hence $[\neg Q(h,k) \vee (\exists z)(z \leq k \wedge S(h,z))]$ is PA-provable for any PA-numeral $[k]$.

(vii) Hence $[(Q(h,k) \rightarrow (\exists z)(z \leq k \wedge S(h,z))]$ is PA-provable for any PA-numeral $[k]$.

(viii) Now, (vii) contradicts our assumption that $[\neg((\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z))))]$ is PA-provable.

(ix) Hence $[\neg((\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z))))]$ is not PA-provable if PA is simply consistent.

However the claimed contradiction in (viii) *only* follows if we assume that PA is ω-consistent, and *not* if we assume that PA is simply consistent.

The presumption of ω-consistency is masked in Rosser's informal proof. It lies in his argument that if, for any given natural number n, the P-formula whose Gödel-number is:

$$Neg(Sb(r \genfrac{}{}{0pt}{}{u}{Z(n)} \genfrac{}{}{0pt}{}{v}{Z(a)})),$$

is P_κ-provable under the given premises, we may conclude that, if P is simply consistent, then the 'generalised' P-formula whose Gödel-number is:

$$uGen(Neg(Sb(r \genfrac{}{}{0pt}{}{v}{Z(a)}))$$

is also P_κ-provable ([Ro36], p90)!

However, a formal proof of this deduction, as above, shows that it presumes P is ω-consistent.

References

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